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editorial

Mathematics – Is it an art, science or Metaphysics?

While calculation remained the core of Mathematics in the early days, as one started learning higher mathematics, mathematics had developed into many branches. Symbolic representation developed into algebra. Study of Euclid's geometry in the westernised studies in the early days developed to co-ordinate geometry and analytical geometry. The study of symmetry in crystals and molecules developed to crystallography and group theory.

In modern studies of diffraction of crystals, group theory is very important. Now, quantum mechanics embraces everything from the determination of energy levels, intensities and widths of spectral lines. If atomic and molecular physicists study X - ray diffraction, nuclear physicists study spectroscopy and diffraction of γ - rays. When mathematics is so interesting and exciting, why do many students and even professors try to avoid mathematics?

It is a purely psychological problem and some fear of punishment. The solution is novelty in packaging which should be so attractive that persons rush to study maths in schools. This is being done in many innovative modern schools. However, to make it accessible to everybody, the solution is simple – to publish popular books in science, mathematics, languages and so on. The key for making education a success is to publish popular books in a simple language so that all can understand these books on self studies on every topic. It is a success for European languages. Why not science and mathematics for every topic written in a popular style?

Anil Ahlawat

Editor

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MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today with the suggestion of Shri Mahabir Singh. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material. During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.
Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

Prof. Dr. Ramanaiah Gundala, Former Dean of Science and Humanities, Anna University, Chennai

PROBLEM Set 147

JEE MAIN

1. In triangle ABC, if $\Delta = a^2 - (b-c)^2$, then $\sin A =$

(a) $\frac{15}{22}$ (b) $\frac{4}{11}$ (c) $\frac{15}{17}$ (d) $\frac{8}{17}$

2. If $y(x) = \frac{x}{1-x^4}$, then the sum of the digits of $\frac{d^5 y(0)}{dx^5}$ is

(a) 3 (b) 4 (c) 5 (d) 0

3. The planes $x + y = 1$, $y - z = 2$, $z + x = 3$ form a triangular prism with cross sectional area

(a) $\frac{4}{\sqrt{3}}$ (b) $\frac{8}{\sqrt{3}}$ (c) $4\sqrt{3}$ (d) $2\sqrt{3}$

4. The area of the triangle, whose vertices are the roots of the equation $x^3 + ix^2 + 2i = 0$, is

(a) 2 (b) 3 (c) $\sqrt{5}$ (d) $\sqrt{7}$

5. $\int_0^{\pi} x^2 \sin 6x\, dx =$

(a) $\frac{\pi^2}{3}$ (b) $-\frac{\pi^2}{3}$ (c) $\frac{\pi^2}{6}$ (d) $-\frac{\pi^2}{6}$

JEE ADVANCED

6. If $C_r = \binom{10}{r}$ and $\frac{C_0}{10} - \frac{C_1}{11} + \frac{C_2}{12} - \frac{C_3}{13} + \ldots + \frac{C_{10}}{20} = \frac{1}{n}$, then n is divisible by

(a) 11 (b) 13 (c) 17 (d) 19

COMPREHENSION

Let $S = \{1, 2, 3, 4, \ldots, 25\}$ and $T = \{x, y\} \subset S$.

7. The probability that $x^2 - y^2$ is divisible by 5 is

(a) $\frac{1}{5}$ (b) $\frac{1}{4}$ (c) $\frac{1}{3}$ (d) $\frac{2}{5}$

8. The probability that $x^2 - y^2$ is divisible by 7 is

(a) $\frac{1}{3}$ (b) $\frac{7}{30}$ (c) $\frac{71}{300}$ (d) $\frac{73}{300}$

INTEGER MATCH

9. Let $8^8 - \binom{8}{1}7^8 + \binom{8}{2}6^8 - \binom{8}{3}5^8 + \binom{8}{4}4^8 - \binom{8}{5}3^8 + \binom{8}{6}2^8 - \binom{8}{7}1^8 = n$. The sum of the digits of n is

MATCHING LIST

10. The sequence of positive integers $a_1, a_2, a_3, \ldots$ is such that a_1, a_2, a_3 are in G.P., a_2, a_3, a_4 are in A.P., a_3, a_4, a_5 are in G.P., a_4, a_5, a_6 are in A.P. etc. Let $a_1 = 1$ and $a_5 + a_6 = 198$. Then match the following:

	Column-I		Column-II
P.	Sum of the digits of a_8 is	1.	5
Q.	Sum of the digits of a_9 is	2.	9
R.	Sum of the digits of a_{10} is	3.	15
S.	Sum of the digits of a_{11} is	4.	19

	P	Q	R	S
(a)	3	2	1	4
(b)	2	1	4	3
(c)	1	4	3	2
(d)	4	3	2	1

See Solution set of Maths Musing 146 on page no. 82

Prof. Ramanaiah is the author of MTG JEE(Main & Advanced) Mathematics series

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MOCK TEST PAPER 2015

JEE Main

Exam on 4th April

1. $\sum_{r=1}^{\infty}(r^2-r+3)x^{r-1}=$

(a) $3+2x(1-x)^{-2}$ (b) $\frac{3x+2}{(1-x)^3}$

(c) $\frac{3(x^2+1)-4x}{(1-x)^3}$ (d) none of these

2. The tangent at (1, 7) to the curve $x^2 = y - 6$ touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at

(a) (6, 7) (b) (–6, 7)

(c) (6, –7) (d) (–6, –7)

3. If $\log_{\sqrt{3}}\left(\frac{|z|^2-|z|+1}{2+|z|}\right)<2$, then the locus of z is

(a) $|z| = 5$ (b) $|z| < 5$

(c) $|z| > 5$ (d) None of these

4. Let $f(x)$ be differentiable on the interval $(0, \infty)$ such that $f(1) = 1$, and $\lim_{t\to x}\frac{t^2f(x)-x^2f(t)}{t-x}=1$ for each $x > 0$. Then $f(x)$ is

(a) $\frac{1}{3x}+\frac{2x^2}{3}$ (b) $-\frac{1}{3x}+\frac{4x^2}{3}$

(c) $-\frac{1}{x}+\frac{2}{x^2}$ (d) $\frac{1}{x}$

5. $\lim_{x\to a}\left(2-\frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}=$

(a) $2/\pi$ (b) $e^{2/\pi}$

(c) $e^{-2/\pi}$ (d) none of these

6. A bag contains 8 red and 5 white balls. Three balls are drawn at random. Find the probability that

I. All the three balls are white

II. All the three balls are red

III. One ball is red and two balls are white

	I	II	III
(a)	$\frac{5}{143}$	$\frac{29}{143}$	$\frac{40}{143}$
(b)	$\frac{5}{143}$	$\frac{28}{143}$	$\frac{40}{143}$
(c)	$\frac{7}{143}$	$\frac{28}{143}$	$\frac{40}{143}$

(d) None of these

7. Let $f : (-1, 1) \to B$, be a function defined by $f(x)=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$, then f is both one-one and onto when B is in the interval

(a) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ (b) $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

(c) $\left[0,\frac{\pi}{2}\right)$ (d) $\left(0,\frac{\pi}{2}\right)$

8. The locus of the centre of circle which touches $(y - 1)^2 + x^2 = 1$ externally and also touches x-axis, is

(a) $\{x^2 = 4y, y \geq 0\} \cup \{(0, y), y < 0\}$

(b) $x^2 = y$

(c) $y = 4x^2$

(d) $\{y^2 = 4x\} \cup (0, y), y \in R$

9. If $\left(\frac{y}{z}\right)^a\left(\frac{z}{x}\right)^b\left(\frac{x}{y}\right)^c=1$

and $A=\left(\frac{y}{z}\right)^{\frac{1}{b-c}}$, $B=\left(\frac{z}{x}\right)^{\frac{1}{c-a}}$, $C=\left(\frac{x}{y}\right)^{\frac{1}{a-b}}$, then

(a) $A = B = C$

(b) $ABC = 1$

(c) $A + B + C = 0$

(d) none of these

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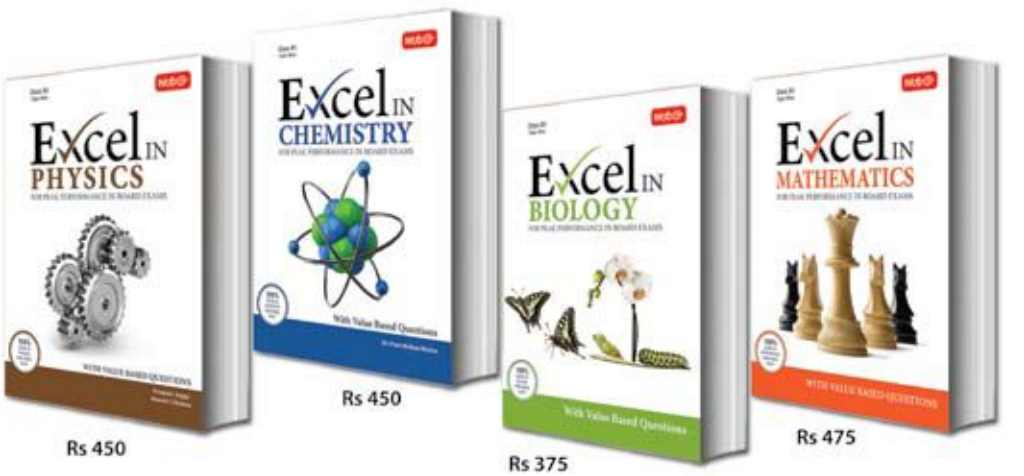

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10. If $\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0,$

then θ is equal to

(a) $\frac{7\pi}{24}, \frac{11\pi}{24}$ (b) $\frac{5\pi}{24}, \frac{7\pi}{24}$

(c) $\frac{11\pi}{24}, \frac{\pi}{24}$ (d) $\frac{\pi}{24}, \frac{7\pi}{24}$

11. If a, b, c are the sides of a ΔABC such that $x^2 - 2(a + b + c)x + 3\lambda(ab + bc + ca) = 0$ has real roots, then

(a) $\lambda < \frac{4}{3}$ (b) $\lambda > \frac{5}{3}$

(c) $\lambda \in \left(\frac{4}{3}, \frac{5}{3}\right)$ (d) $\lambda \in \left(\frac{1}{3}, \frac{5}{3}\right)$

12. The expression

$$3\left[\sin^4\left(\frac{3\pi}{2} - \alpha\right) + \sin^4(3\pi + \alpha)\right]$$

$$-2\left[\sin^6\left(\frac{\pi}{2} + \alpha\right) + \sin^6(5\pi - \alpha)\right] \text{ is equal to}$$

(a) 0 (b) 1

(c) 3 (d) $\sin 4\alpha + \cos 6\alpha$

13. 5-digit numbers are to be formed using 2, 3, 5, 7, 9 without repeating the digits. If p be the number of such numbers that exceed 20000 and q be the number of those that lie between 30000 and 90000, then $p : q$ is

(a) 6 : 5 (b) 3 : 2

(c) 4 : 3 (d) 5 : 3

14. The area bounded by the curves $y = (x - 1)^2$, $y = (x + 1)^2$ and $y = \frac{1}{4}$ is

(a) $\frac{1}{3}$ sq. unit (b) $\frac{2}{3}$ sq. unit

(c) $\frac{1}{4}$ sq. unit (d) $\frac{1}{5}$ sq. unit

15. Given $\vec{a} = 2\hat{i} + \hat{j} - 2\hat{k}$, $\vec{b} = 2\hat{i} + \hat{j}$, $|\vec{c}| = \vec{a} \cdot \vec{c}$, $|\vec{c} - \vec{a}| = 4\cos\frac{\pi}{4}$ and angle between $\vec{c}$ & $(\vec{a} \times \vec{b})$ is $\pi/6$, then $|(\vec{a} \times \vec{b}) \times \vec{c}|$ is equal to

(a) $\sqrt{5}$ (b) 1

(c) 3/2 (d) none of these

16. If $\tan^{-1} x + \sin^{-1} x = \frac{\pi}{2}$, then $x =$

(a) $\pm\sqrt{\frac{\sqrt{5} - 1}{2}}$ (b) $\pm\sqrt{\frac{\sqrt{5} \pm 1}{2}}$

(c) $\pm\sqrt{\frac{\sqrt{5} + 1}{2}}$ (d) none of these

17. For $0 \le x \le \frac{\pi}{2}$, the value of

$\int_0^{\sin^2 x} \sin^{-1}(\sqrt{t})dt + \int_0^{\cos^2 x} \cos^{-1}(\sqrt{t})dt$ is equal to

(a) $-\frac{\pi}{4}$ (b) 0 (c) 1 (d) $\frac{\pi}{4}$

18. For each real $x : -1 < x < 1$. Let $A(x)$ be the matrix $(1 - x)^{-1}\begin{bmatrix} 1 & -x \\ -x & 1 \end{bmatrix}$ and $z = \frac{x + y}{1 + xy}$, then

(a) $A(z) = A(x) - A(y)$

(b) $A(z) = A(x)\, A(y)$

(c) $A(z) = A(x)\, [A(y)]^{-1}$

(d) $A(z) = A(x) + A(y)$

19. A variable plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ at a unit distance from origin cuts the coordinate axes at A, B and C. Centroid (x, y, z) satisfies the equation $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = K$. The value of K is

(a) 9 (b) 3 (c) $\frac{1}{9}$ (d) $\frac{1}{3}$

20. If a_r be the coefficient of x_r in the expansion of $(1 - x)^{2011}$, then sum of which of the following pair vanishes?

(a) a_{777}, a_{1234} (b) a_{1111}, a_{900}

(c) a_{654}, a_{1357} (d) All of these

21. The value of $\int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1 + 2^x} dx$ is

(a) π (b) $\frac{\pi}{2}$

(c) 4π (d) $\frac{\pi}{4}$

22. The locus of the orthocentre of the triangle formed by the lines $(1 + p)x - py + p(1 + p) = 0$, $(1 + q)x - qy + q(1 + q) = 0$ and $y = 0$, where $p \neq q$, is

(a) a hyperbola (b) a parabola

(c) an ellipse (d) a straight line

23. If a variable x takes values x_i such that $a \leq x_i \leq b$, for $i = 1, 2, ..., n$, then

(a) $a^2 \leq \text{var}(x) \leq b^2$ (b) $a \leq \text{var}(x) \leq b$

(c) $\frac{a^2}{4} \leq \text{var}(x)$ (d) $(b - a)^2 \geq \text{var}(x)$

24. If $\frac{3}{2 + e^{i\theta}} = \lambda x + i\mu y$, then locus of $P(x, y)$ will represent a/an

(a) Ellipse if $\lambda = 1$, $\mu = 2$

(b) Pair of straight lines if $\mu = 1$, $\lambda = 0$

(c) Circle if $\lambda \neq \mu \neq 1$

(d) None of these

25. Domain of definition of the function

$$f(x) = \frac{3}{4 - x^2} + \log_{10}(x^3 - x), \text{ is}$$

(a) $(1, 2)$

(b) $(-1, 0) \cup (1, 2)$

(c) $(1, 2) \cup (2, \infty)$

(d) $(-1, 0) \cup (1, 2) \cup (2, \infty)$

26. The set of all points, where the function $f(x) = \frac{x}{1 + |x|}$ is differentiable, is

(a) $(-\infty, \infty)$ (b) $[0, \infty)$

(c) $(-\infty, 0) \cup (0, \infty)$ (d) $(0, \infty)$

27. A particle is projected vertically upward and reaches at a height of h after time t seconds. It further takes t' seconds to reach the ground. Let the greatest height attained be H, then

(a) A.M. of t and $t' = \frac{1}{g}$

(b) G.M. of t and $t' = \sqrt{\frac{2h}{g}}$

(c) A.M. of t and $t' = \sqrt{\frac{2h}{g}}$

(d) None of these

28. Tangent is drawn to ellipse $\frac{x^2}{27} + y^2 = 1$ at $(3\sqrt{3}\cos\theta, \sin\theta)$ (where $\theta \in (0, \pi/2)$). Then, the value of θ such that the sum of intercepts on axes made by this tangent is minimum, is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{8}$ (d) $\frac{\pi}{4}$

29. The complex numbers $z = x + iy$ which satisfy the equation $\left|\frac{z - 5i}{z + 5i}\right| = 1$, lie on

(a) x-axis

(b) straight line $y = 5$

(c) a circle passing through the origin

(d) None of these

30. If x_1, x_2, x_3 and x_4 are the roots of the equation $x^4 - x^3 \sin 2\beta + x^2 \cos 2\beta - x \cos\beta - \sin\beta = 0$, then $\tan^{-1} x_1 + \tan^{-1} x_2 + \tan^{-1} x_3 + \tan^{-1} x_4$ is equal to

(a) β (b) $\frac{\pi}{2} - \beta$

(c) $\pi - \beta$ (d) $-\beta$

SOLUTIONS

1. (c) : Let $S = \sum_{r=1}^{\infty} (r^2 - r + 3)x^{r-1}$

$\therefore \quad S = 3 + 5x + 9x^2 + 15x^3 + ... \text{ to } \infty$

$$(-)\ \underline{x \cdot S = \quad 3x + 5x^2 + 9x^3 + ... \text{ to } \infty}$$

$$(1 - x)S = 3 + 2x + 4x^2 + 6x^3 + ... \text{ to } \infty$$

$$= 3 + 2x(1 + 2x + 3x^2 + ... \text{ to } \infty)$$

$$= 3 + 2x(1 - x)^{-2} = \frac{3(1 - x)^2 + 2x}{(1 - x)^2} = \frac{3(x^2 + 1) - 4x}{(1 - x)^2}$$

$$\therefore \quad S = \frac{3(x^2 + 1) - 4x}{(1 - x)^3}$$

2. (d) : The tangent to the parabola $x^2 = y - 6$ at $(1, 7)$ is $y = 2x + 5$

which is also a tangent to the given circle.

i.e., $x^2 + (2x + 5)^2 + 16x + 12(2x + 5) + c = 0$

$\Rightarrow 5x^2 + 60x + 85 + c = 0$ must have equal roots.

Let the roots be α, α

$\therefore \quad \alpha + \alpha = -\frac{60}{5} \Rightarrow \alpha = -6$

$\therefore \quad x = -6$ and $y = 2x + 5 = -7$

3. (b) : $\log_{\sqrt{3}}\left(\frac{|z|^2 - |z| + 1}{2 + |z|}\right) < 2$

$\Rightarrow \quad \frac{|z|^2 - |z| + 1}{2 + |z|} < (\sqrt{3})^2$

$\Rightarrow \quad |z|^2 - |z| + 1 < 3(2 + |z|)$

$\Rightarrow \quad |z|^2 - 4|z| - 5 < 0$

$\Rightarrow \quad (|z| + 1)(|z| - 5) < 0 \Rightarrow -1 < |z| < 5$

$\Rightarrow \quad |z| < 5$ as $|z| > 0$

$\therefore \quad$ Locus of z is $|z| < 5$.

4. (a) : Given, $\lim_{t \to x} \frac{t^2 f(x) - x^2 f(t)}{t - x} = 1$

$\Rightarrow \quad x^2 f'(x) - 2xf(x) + 1 = 0$

$\Rightarrow \quad \frac{x^2 f'(x) - 2xf(x)}{(x^2)^2} + \frac{1}{x^4} = 0$

$\Rightarrow \quad \frac{d}{dx}\left(\frac{f(x)}{x^2}\right) = -\frac{1}{x^4}$

On integrating both sides, we get

$f(x) = cx^2 + \frac{1}{3x}$

Also $f(1) = 1 \Rightarrow c = \frac{2}{3}$

Hence $f(x) = \frac{2}{3}x^2 + \frac{1}{3x}$

5. (b) : $\because$ Given limit is in 1^∞ form

$\therefore \quad \lim_{x \to a}\left(2 - \frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$

$= e^{\lim_{x \to a}\left(2 - \frac{x}{a} - 1\right)\tan\left(\frac{\pi x}{2a}\right)} = e^{\lim_{x \to a} \frac{\left(1 - \frac{x}{a}\right)}{\cot\left(\frac{\pi x}{2a}\right)}}$

$= e^{\lim_{x \to a} \frac{\left(-\frac{1}{a}\right)}{-\text{cosec}^2\left(\frac{\pi x}{2a}\right)\frac{\pi}{2a}}}$ [L-Hospital's Rule]

$= e^{\lim_{x \to a} \frac{2}{\pi}\sin^2\left(\frac{\pi x}{2a}\right)} = e^{2/\pi}$

6. (b) : I. P(All three balls are white) $= \frac{{}^5C_3}{{}^{13}C_3}$

$= \frac{5! \times 10!}{2! \times 13!} = \frac{5 \times 4 \times 3}{13 \times 12 \times 11} = \frac{5}{143}$

II. P(All the three balls are red) $= \frac{{}^8C_3}{{}^{13}C_3}$

$= \frac{8! \times 10!}{5! \times 13!} = \frac{8 \times 7 \times 6}{13 \times 12 \times 11} = \frac{28}{143}$

III. P(One ball is red and two balls are white)

$= \frac{{}^8C_1 \times {}^5C_2}{{}^{13}C_3} = \frac{8 \times 10}{\frac{13 \times 12 \times 11}{3 \times 2}} = \frac{40}{143}$

7. (a) : Since, $x \in (-1, 1) \Rightarrow \tan^{-1} x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

$\Rightarrow \quad 2\tan^{-1} x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

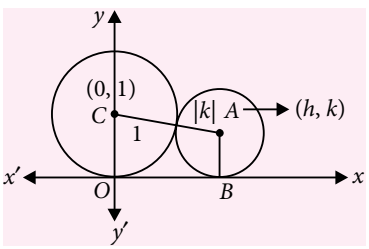
Given that,

$f(x) = \tan^{-1}\left(\frac{2x}{1 - x^2}\right) = 2\tan^{-1} x \quad (\because \ x^2 < 1)$

So, $f(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Hence, function is one-one onto.

8. (a) : Let the locus of centre of circle be (h, k) touching $(y - 1)^2 + x^2 = 1$ and x-axis shown as



distance between O and A is always $1 + |k|$,

i.e., $\sqrt{(h - 0)^2 + (k - 1)^2} = 1 + |k|$,

$\Rightarrow \quad h^2 + k^2 - 2k + 1 = 1 + k^2 + 2|k|$

$\Rightarrow \quad h^2 = 2|k| + 2k$

$\Rightarrow \quad x^2 = 2|y| + 2y$

where, $|y| = \begin{cases} y, & y \geq 0 \\ -y, & y < 0 \end{cases}$

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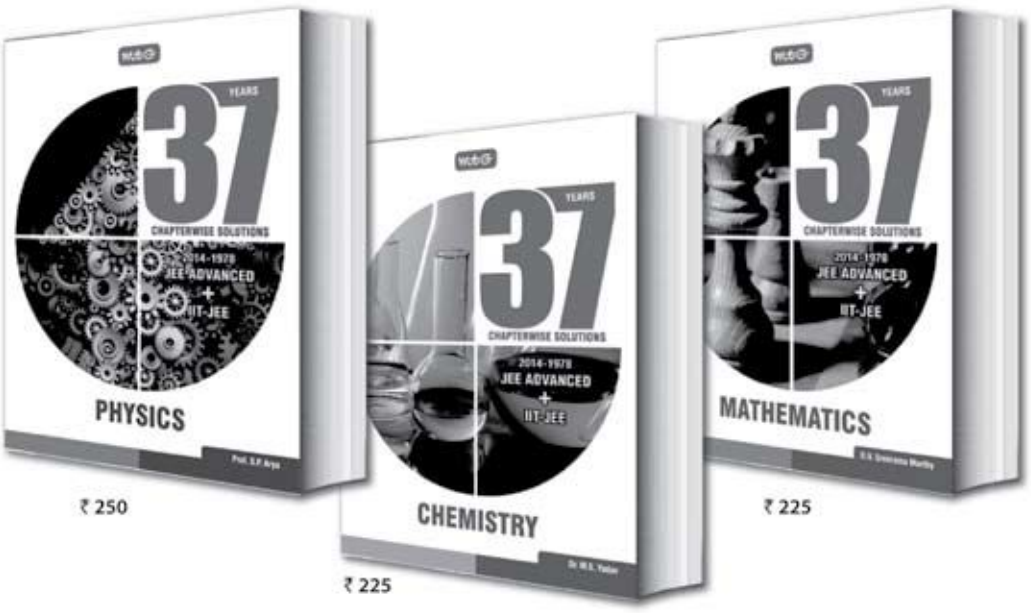

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$\therefore \quad x^2 = 2y + 2y,\ y \geq 0$ and $x^2 = -2y + 2y,\ y < 0$

$\Rightarrow \quad x^2 = 4y$ when $y \geq 0$ and $x^2 = 0$ when $y < 0$

$\therefore \quad \{(x, y) : x^2 = 4y,\ y \geq 0\} \cup \{(0, y),\ y < 0\}$

9. (a) : We have,

$$\left(\frac{z}{x}\right)^b \cdot \left(\frac{x}{y}\right)^c = \left(\frac{z}{y}\right)^a = \left(\frac{z}{x} \cdot \frac{x}{y}\right)^a = \left(\frac{z}{x}\right)^a \cdot \left(\frac{x}{y}\right)^a$$

$$\Rightarrow \left(\frac{x}{y}\right)^{c-a} = \left(\frac{z}{x}\right)^{a-b}$$

$$\Rightarrow \left(\frac{x}{y}\right)^{\frac{1}{a-b}} = \left(\frac{z}{x}\right)^{\frac{1}{c-a}} \Rightarrow C = B$$

Similarly, it can be proved that $A = B$

$\therefore \quad A = B = C.$

10. (a) : Applying $R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$ we get,

$$\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ \sin^2\theta & \cos^2\theta & 1 + 4\sin 4\theta \end{vmatrix} = 0$$

$\Rightarrow \quad \sin^2\theta + \cos^2\theta + 1 + 4\sin 4\theta = 0$

$$\Rightarrow \quad \sin 4\theta = -\frac{1}{2} \quad \Rightarrow \quad \theta = \frac{7\pi}{24}, \frac{11\pi}{24}$$

11. (a) : Since, roots are real, therefore $D \geq 0$.

$\Rightarrow \quad 4(a + b + c)^2 - 12\lambda\,(ab + bc + ca) \geq 0$

$\Rightarrow \quad (a + b + c)^2 \geq 3\lambda\,(ab + bc + ca)$

$\Rightarrow \quad a^2 + b^2 + c^2 \geq (ab + bc + ca)\,(3\lambda - 2)$

$$\Rightarrow \quad 3\lambda - 2 \leq \frac{a^2 + b^2 + c^2}{ab + bc + ca} \qquad \text{...(i)}$$

Also, $\quad \cos A = \dfrac{b^2 + c^2 - a^2}{2bc} < 1$

$\Rightarrow \quad b^2 + c^2 - a^2 < 2bc$

Similarly, $c^2 + a^2 - b^2 < 2ca$

and $a^2 + b^2 - c^2 < 2ab$

$\Rightarrow \quad a^2 + b^2 + c^2 < 2(ab + bc + ca)$

$$\Rightarrow \quad \frac{a^2 + b^2 + c^2}{ab + bc + ca} < 2 \qquad \text{...(ii)}$$

$\therefore \quad$ From (i) and (ii), we get

$$3\lambda - 2 < 2 \Rightarrow \quad \lambda < \frac{4}{3}$$

12. (b) : $3\left[\sin^4\left(\dfrac{3\pi}{2} - \alpha\right) + \sin^4(3\pi + \alpha)\right]$

$$-2\left[\sin^6\left(\frac{\pi}{2} + \alpha\right) + \sin^6(5\pi - \alpha)\right]$$

$= 3(\cos^4\alpha + \sin^4\alpha) - 2(\cos^6\alpha + \sin^6\alpha)$

$= 3(1 - 2\sin^2\alpha\cos^2\alpha) - 2(1 - 3\sin^2\alpha\cos^2\alpha)$

$= 3 - 6\sin^2\alpha\cos^2\alpha - 2 + 6\sin^2\alpha\cos^2\alpha = 1$

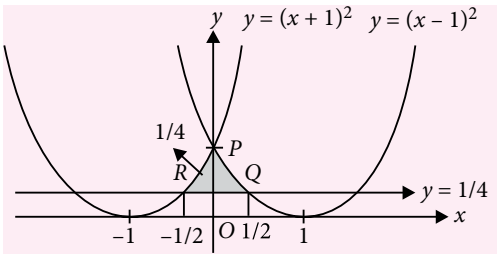
13. (d) : p = The number of such numbers that exceeds 20000 $= 5! = 120$

q = The number of those that lie between 30000 and 90000

$= 5! - 4! - 4! = 120 - 24 - 24 = 72$

$$\therefore \quad \frac{p}{q} = \frac{120}{72} = \frac{5}{3}$$

14. (a) : The curves $y = (x - 1)^2$, $y = (x + 1)^2$ and $y = 1/4$ are shown as



where point of intersection are

$$(x - 1)^2 = \frac{1}{4} \quad \Rightarrow \quad x = \frac{1}{2}$$

and $\quad (x + 1)^2 = \dfrac{1}{4} \quad \Rightarrow \quad x = -\dfrac{1}{2}$

$$\therefore \quad Q\left(\frac{1}{2}, \frac{1}{4}\right) \text{ and } R\left(-\frac{1}{2}, \frac{1}{4}\right)$$

$$\therefore \quad \text{Required area} = 2\int_0^{1/2}\left[(x-1)^2 - \frac{1}{4}\right]dx$$

$$= 2\left[\frac{(x-1)^3}{3} - \frac{1}{4}x\right]_0^{1/2}$$

$$= 2\left[-\frac{1}{8\cdot 3} - \frac{1}{8} - \left(-\frac{1}{3} - 0\right)\right] = \frac{1}{3}\text{ sq. unit}$$

15. (a) : $|\vec{c}-\vec{a}|=4\cos\frac{\pi}{4}=2\sqrt{2} \Rightarrow |\vec{c}-\vec{a}|^2=8$

$\Rightarrow (\vec{c}-\vec{a})\cdot(\vec{c}-\vec{a})=8$

$\Rightarrow |\vec{c}|^2+|\vec{a}|^2-2(\vec{a}\cdot\vec{c})=8$

$\Rightarrow |\vec{c}|^2-2|\vec{c}|=8-9 \qquad [\because\ |\vec{a}|=3]$

$\Rightarrow |\vec{c}|=1$

Now, $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\2&1&-2\\2&1&0\end{vmatrix}=2\hat{i}-4\hat{j}$

$\Rightarrow |\vec{a}\times\vec{b}|=\sqrt{20}$

$\therefore\ |(\vec{a}\times\vec{b})\times\vec{c}|=|\vec{a}\times\vec{b}||\vec{c}|\sin\frac{\pi}{6}$

$=\sqrt{20}(1)\left(\frac{1}{2}\right)=\sqrt{5}$

16. (d) : We have, $\tan^{-1}x=\frac{\pi}{2}-\sin^{-1}x=\cos^{-1}x$

$\Rightarrow \sec^{-1}\sqrt{1+x^2}=\sec^{-1}\left(\frac{1}{x}\right)$

$\Rightarrow 1+x^2=\frac{1}{x^2} \Rightarrow x^4+x^2-1=0$

$\therefore\ x^2=\frac{-1\pm\sqrt{1-4\cdot1\cdot(-1)}}{2}=\frac{-1\pm\sqrt{5}}{2}$

$\because\ x^2\neq0\ \ \therefore\ x^2=\frac{\sqrt{5}-1}{2}$

$\therefore\ x=\pm\sqrt{\frac{\sqrt{5}-1}{2}}$

But, if $x<0$ then L.H.S. of given equation becomes –ve and given equation is not satisfied.

$\therefore\ x=\sqrt{\frac{\sqrt{5}-1}{2}}$

17. (d) : Put $t=\sin^2 z$ in 1st integral and $t=\cos^2 u$ in 2nd integral.

$dt=2\sin z\cos z\,dz$ and

$dt=-2\cos u\sin u\,du$

$\therefore\ I=\int_0^x 2z\sin z\cos z\,dz+\int_{\pi/2}^x -2u\cos u\sin u\,du$

$=\int_0^x \underset{\text{I}}{z}\ \underset{\text{II}}{\sin 2z}\,dz-\int_{\pi/2}^x u\sin 2u\,du$

$=\left[-z\frac{\cos2z}{2}+\frac{\sin2z}{4}\right]_0^x-\left[\frac{-u\cos2u}{2}+\frac{\sin2u}{4}\right]_{\pi/2}^x$

$=\left[-x\frac{\cos2x}{2}+\frac{\sin2x}{4}-\{0+0\}\right]$

$-\left[-x\frac{\cos2x}{2}+\frac{\sin2x}{4}-\left(\frac{\pi}{4}+0\right)\right]=\frac{\pi}{4}$

18. (b) : We have, $A(x)\ A(y)$

$=(1-x)^{-1}\begin{bmatrix}1&-x\\-x&1\end{bmatrix}(1-y)^{-1}\begin{bmatrix}1&-y\\-y&1\end{bmatrix}$

$=[(1+xy)-(x+y)]^{-1}\begin{bmatrix}1+xy&-(x+y)\\-(x+y)&1+xy\end{bmatrix}$

$=\left(1-\frac{x+y}{1+xy}\right)^{-1}\begin{bmatrix}1&-\left(\frac{x+y}{1+xy}\right)\\-\left(\frac{x+y}{1+xy}\right)&1\end{bmatrix}$

$=A(z)$

19. (a) : Since, $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ cuts the coordinate axis at $A(a,0,0)$, $B(0,b,0)$, $C(0,0,c)$ and its distance from origin = 1

$\therefore\ \frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}=1$ or $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=1$ ...(i)

Let P be the centroid of triangle

$\therefore\ P(x,y,z)=\left(\frac{a+0+0}{3},\frac{0+b+0}{3},\frac{0+0+c}{3}\right)$

$\Rightarrow\ x=\frac{a}{3},\ y=\frac{b}{3},\ z=\frac{c}{3}$ ...(ii)

$\therefore$ From (i) and (ii), we get

$\frac{1}{9x^2}+\frac{1}{9y^2}+\frac{1}{9z^2}=1 \Rightarrow \frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=9=K$

$\therefore\ K=9$

20. (d) : $\because\ t_{r+1}={}^{2011}C_r(-x)^r=(-1)^r\cdot{}^{2011}C_r\cdot x^r$

$\therefore$ According to problem, $a_r=(-1)^r\cdot{}^{2011}C_r$

Also, $a_{777}+a_{1234}=(-1)^{777}\,{}^{2011}C_{777}+(-1)^{1234}\cdot{}^{2011}C_{1234}$

$=-{}^{2011}C_{777}+{}^{2011}C_{777}=0$

Also, $a_{1111} + a_{900} = (-1)^{1111} \cdot {}^{2011}C_{1111} + (-1)^{900} \cdot {}^{2011}C_{900}$

$= - {}^{2011}C_{1111} + {}^{2011}C_{1111} = 0$

Similarly,

$a_{654} + a_{1357} = (-1)^{654} {}^{2011}C_{654} + (-1)^{1357} {}^{2011}C_{1357}$

$= {}^{2011}C_{654} - {}^{2011}C_{654} = 0$

21. (d) : Let $I = \int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^x} dx$

$\Rightarrow \quad I = \int_{-\pi/2}^{\pi/2} \frac{2^x \sin^2 x}{1+2^x} dx$

$\Rightarrow \quad 2I = \int_{-\pi/2}^{\pi/2} \sin^2 x dx = \int_0^{\pi/2} 2\sin^2 x dx$

$= \int_0^{\pi/2} (1 - \cos 2x) dx$

$\Rightarrow \quad 2I = \left[x - \frac{\sin 2x}{2} \right]_0^{\pi/2} = \frac{\pi}{2} \Rightarrow \quad I = \frac{\pi}{4}$

22. (d) : $(1 + p)x - py + p(1 + p) = 0$ ...(i)

$(1 + q)x - qy + q(1 + q) = 0$ ...(ii)

On solving (i) and (ii), we get

$x = pq$, $y = (1 + p)(1 + q)$

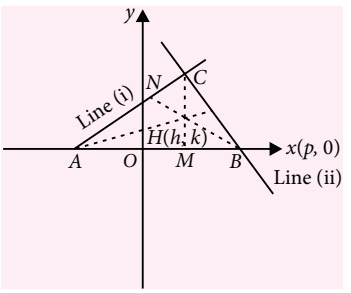
$\therefore$ Coordinates of C are $\{pq, (1 + p)(1 + q)\}$.

$\therefore$ Equation of altitude CM passing through C and perpendicular to AB is

$x = pq$ ...(iii)

$\because$ Slope of line (ii) is $\left(\frac{1+q}{q} \right)$

$\therefore$ Slope of altitude BN (as shown in figure) is $\frac{-q}{1+q}$.



Equation of BN is $y - 0 = \frac{-q}{1+q}(x + p)$

$\Rightarrow \quad y = \frac{-q}{(1+q)}(x + p)$ ...(iv)

Let orthocentre of triangle be $H(h, k)$ which is the point of intersection of (iii) and (iv).

On solving (iii) and (iv), we get

$x = pq$ and $y = -pq$

$\Rightarrow \quad h = pq$ and $k = -pq$

$\therefore \quad h + k = 0$

$\therefore$ Locus of $H(h, k)$ is $x + y = 0$.

23. (d) : Since, S.D. < Range

$\Rightarrow \quad \sigma \le (b - a) \Rightarrow \quad \sigma^2 \le (b - a)^2$

24. (a) : $2 + e^{i\theta} = \frac{3}{\lambda x + i\mu y}$

$\Rightarrow \quad 2 + \cos\theta + i\sin\theta = \frac{3}{\lambda x + i\mu y}$

$\Rightarrow \quad \cos\theta + i\sin\theta = \frac{(3 - 2\lambda x) - 2i\mu y}{\lambda x + i\mu y}$

On taking modulus to both sides,

$\Rightarrow \quad 1 = \frac{\sqrt{(3 - 2\lambda x)^2 + (-2\mu y)^2}}{\sqrt{(\lambda x)^2 + (\mu y)^2}}$

$\Rightarrow \quad \lambda^2 x^2 + \mu^2 y^2 = 9 - 12\lambda x + 4\lambda^2 x^2 + 4\mu^2 y^2$

$\Rightarrow \quad \lambda^2 x^2 + \mu^2 y^2 - 4\lambda x + 3 = 0$ ...(1)

This is the locus of $P(x, y)$

If $\lambda = 1$, $\mu = 2$, then (1) becomes

$x^2 + 4y^2 - 4x + 3 = 0$

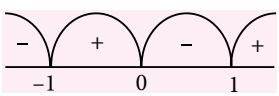
$\Rightarrow \quad (x - 2)^2 + 4y^2 = 1 \Rightarrow \frac{(x-2)^2}{1^2} + \frac{y^2}{1/4} = 1,$

which is an ellipse.

25. (d) : Since, $f(x) = \frac{3}{4 - x^2} + \log_{10}(x^3 - x)$

For domain of $f(x)$,

$x^3 - x > 0 \quad \Rightarrow \quad x(x - 1)(x + 1) > 0$



Region is $(-1, 0) \cup (1, \infty)$

And $4 - x^2 \ne 0$

$\Rightarrow \quad x \ne \pm 2$

Region is $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.

$\therefore$ Common region is $(-1, 0) \cup (1, 2) \cup (2, \infty)$

26. (b) : Given, $f(x) = \dfrac{x}{1+|x|} = \begin{cases} \dfrac{x}{1+x}, & x \geq 0 \\ \dfrac{x}{1-x}, & x < 0 \end{cases}$

$\Rightarrow \quad f'(x) = \begin{cases} \dfrac{1}{(1+x)^2}, & x \geq 0 \\ \dfrac{1}{(1-x)^2}, & x < 0 \end{cases}$

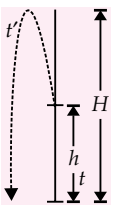
$\therefore$ RHD at $x = 0 \Rightarrow \lim\limits_{x \to 0} \dfrac{1}{(1+x)^2} = 1$

and LHD at $x = 0 \Rightarrow \lim\limits_{x \to 0} \dfrac{1}{(1-x)^2} = 1$

Hence, $f(x)$ is differentiable for all x.

27. (b) :

Since total time $= t + t' = \dfrac{2u}{g}$



$\therefore \quad \dfrac{t+t'}{2} = \dfrac{u}{g}$

$\Rightarrow \quad u = \dfrac{g(t+t')}{2}$ ...(i)

$\because \quad H = \dfrac{u^2}{2g} = \dfrac{1}{2g}\left\{\dfrac{g(t+t')}{2}\right\}^2 = \dfrac{g}{8}(t+t')^2$

$\therefore \quad \dfrac{t+t'}{2} = \sqrt{\dfrac{2H}{g}}$

Also,

$h = ut - \dfrac{1}{2}gt^2 = \dfrac{g(t+t')}{2}t - \dfrac{1}{2}gt^2 = \dfrac{gtt'}{2}$ [From (i)]

$\therefore \quad tt' = \dfrac{2h}{g} \quad \Rightarrow \quad \sqrt{tt'} = \sqrt{\dfrac{2h}{g}}$

28. (b) : Given, tangent is drawn

at $(3\sqrt{3}\cos\theta, \sin\theta)$ to $\dfrac{x^2}{27} + \dfrac{y^2}{1} = 1$.

$\therefore$ Equation of tangent is $\dfrac{x\cos\theta}{3\sqrt{3}} + \dfrac{y\sin\theta}{1} = 1$.

Thus, sum of intercepts

$= \left(\dfrac{3\sqrt{3}}{\cos\theta} + \dfrac{1}{\sin\theta}\right) = f(\theta)$ (say)

$\Rightarrow \quad f'(\theta) = \dfrac{3\sqrt{3}\sin^3\theta - \cos^3\theta}{\sin^2\theta\cos^2\theta}$

For maxima/minima $f'(\theta) = 0$

$\Rightarrow \quad \sin^3\theta = \dfrac{1}{3^{3/2}}\cos^3\theta$

$\Rightarrow \quad \tan\theta = \dfrac{1}{\sqrt{3}}, \ i.e., \ \theta = \dfrac{\pi}{6}$

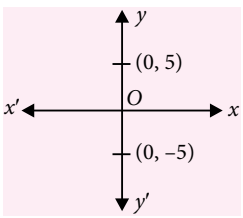
and at $\theta = \dfrac{\pi}{6}$, $f''(0) > 0$,

$\therefore$ Hence, tangent is minimum at $\theta = \dfrac{\pi}{6}$.

29. (a) : Given, $\left|\dfrac{z-5i}{z+5i}\right| = 1$

$\Rightarrow \quad |z - 5i| = |z + 5i|$

(if $|z - z_1| = |z - z_2|$, then it is a perpendicular bisector of z_1 and z_2)



$\therefore$ Perpendicular bisector of (0, 5) and (0, –5) is x-axis.

30. (b) : We have, $\Sigma x_1 = \sin 2\beta$, $\Sigma x_1x_2 = \cos 2\beta$, $\Sigma x_1x_2x_3 = \cos\beta$ and $x_1x_2x_3x_4 = -\sin\beta$

$\therefore \quad \tan^{-1}x_1 + \tan^{-1}x_2 + \tan^{-1}x_3 + \tan^{-1}x_4$

$= \tan^{-1}\left(\dfrac{\Sigma x_1 - \Sigma x_1x_2x_3}{1 - \Sigma x_1x_2 + x_1x_2x_3x_4}\right)$

$= \tan^{-1}\left(\dfrac{\sin 2\beta - \cos\beta}{1 - \cos 2\beta - \sin\beta}\right)$

$= \tan^{-1}\left[\dfrac{(2\sin\beta - 1)\cos\beta}{\sin\beta(2\sin\beta - 1)}\right] = \tan^{-1}(\cot\beta)$

$= \tan^{-1}\left\{\tan\left(\dfrac{\pi}{2} - \beta\right)\right\} = \dfrac{\pi}{2} - \beta$

PRACTICE PAPER 2015

BITSAT

Exam on 14th to 29th May

1. If $f(x)=\frac{x}{x-1}, x\neq 1$, then $\underbrace{(fofo.....of)(x)}_{19 \text{ times}}$ is equal to

(a) $\frac{x}{x-1}$ (b) $\left(\frac{x}{x-1}\right)^{19}$

(c) $\frac{19x}{x-1}$ (d) x

2. If a, b, c are in H.P., b, c, d are in G.P. and c, d, e are in A.P., then $\frac{ab^2}{(2a-b)^2}$ is equal to

(a) b (b) a (c) e (d) d

3. The imaginary part of $(z-1)(\cos\alpha - i\sin\alpha) + (z-1)^{-1}(\cos\alpha + i\sin\alpha)$ is zero, if

(a) $|z-1| = 2$ (b) $\arg(z-1) = 2\alpha$

(c) $\arg(z-1) = \alpha$ (d) $|z| = 1$

4. The values of 'a' for which $(a^2-1)x^2 + 2(a-1)x + 2$ is positive for any x, are

(a) $a > 1$ (b) $a \leq 1$

(c) $a > -3$ (d) $a < 1$

5. If $\cos\theta = \frac{-3}{5}$ and $\pi < \theta < \frac{3\pi}{2}$, then the value of $\frac{\text{cosec}\theta + \cot\theta}{\sec\theta - \tan\theta}$ is

(a) 1/6 (b) 1/7 (c) 1/5 (d) 1/2

6. Solution of $(2x+1)(x-3)(x+7) < 0$ is

(a) $(-\infty, -7) \cup \left(-\frac{1}{2}, 3\right)$

(b) $(-\infty, -7) \cup \left(\frac{1}{2}, 3\right)$

(c) $(-\infty, 7) \cup \left(-\frac{1}{2}, 3\right)$

(d) $(-\infty, -7) \cup (3, \infty)$

7. How many numbers lying between 100 and 1000 can be formed with the digits 0, 1, 2, 3, 4, 5, if the repetition of the digits is not allowed?

(a) 90 (b) 50 (c) 40 (d) 100

8. Assuming that straight line work as the plane mirror for a point, find the image of the point (1, 2) in the line $x - 3y + 4 = 0$.

(a) (–2, –1) (b) (–1, –2)

(c) $\left(\frac{6}{5}, \frac{7}{5}\right)$ (d) $\left(\frac{-6}{5}, \frac{-7}{5}\right)$

9. The equation $\frac{x^2}{14-a} + \frac{y^2}{9-a} = 1$ represents a/an

(a) ellipse if $a > 9$

(b) hyperbola if $9 < a < 14$

(c) hyperbola if $a > 14$

(d) ellipse if $9 < a < 14$

10. $\lim_{x\to 0} \frac{2\sin x - \sin 2x}{x^3}$ is equal to

(a) –1 (b) 1 (c) 2 (d) –2

11. For any vector $\vec{a}$, prove that $|\vec{a}\times\hat{i}|^2 + |\vec{a}\times\hat{j}|^2 + |\vec{a}\times\hat{k}|^2$ is equal to

(a) $2|\vec{a}|^2$ (b) $|\vec{a}|^2$ (c) $3|\vec{a}|^2$ (d) $4|\vec{a}|^2$

12. If the scalar product of the vector $\hat{i}+\hat{j}+\hat{k}$ with the unit vector in the direction of the resultant of the vectors $2\hat{i}+4\hat{j}-5\hat{k}$ and $\lambda\hat{i}+2\hat{j}+3\hat{k}$ is unity, then λ equals

(a) 5 (b) 2 (c) 1 (d) –1

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13. The first 12 letters of English alphabet are written in a row at random. The probability that there are exactly four letters in between A and B is

(a) $\frac{5}{66}$ (b) $\frac{1}{22}$ (c) $\frac{7}{66}$ (d) $\frac{1}{11}$

14. The probability of the birth dates of all 6 persons to fall in only two different months is

(a) $\frac{341}{12^6}$ (b) $\frac{341}{12^5}$ (c) $\frac{341}{12^4}$ (d) $\frac{541}{12^6}$

15. If x, y, z are positive real numbers and $x+y+z=1$, then prove that the minimum value of $\frac{4}{x}+\frac{9}{y}+\frac{16}{z}$ is

(a) 80 (b) 81 (c) 85 (d) 82

16. In ΔABC, if $8R^2 = a^2 + b^2 + c^2$, then the triangle is

(a) isosceles (b) right angled
(c) equilateral (d) scalene

17. The number of solutions of the equation $1+\sin x \sin^2\frac{x}{2}=0$ in $[-\pi, \pi]$ is

(a) zero (b) one (c) two (d) three

18. $\int_0^2 [x^2]dx$ is equal to

(a) $2-\sqrt{2}$ (b) $2+\sqrt{2}$
(c) $\sqrt{2}-1$ (d) $-\sqrt{2}-\sqrt{3}+5$

19. Let $f: (2, 3) \to (0, 1)$ be defined by $f(x) = x - [x]$. Then, $f^{-1}(x)$ equals to

(a) $x-2$ (b) $x+1$ (c) $x-1$ (d) $x+2$

20. Which of the following is/are true?

(i) The principal value of $\cos^{-1}\frac{\sqrt{3}}{2}$ is $\frac{\pi}{6}$.

(ii) The principal value of $\text{cosec}^{-1}(2)$ is $\frac{\pi}{4}$.

(iii) The principal value of $\tan^{-1}(-\sqrt{3})$ is $\frac{-\pi}{3}$.

(a) (i), (ii) (b) (ii), (iii)
(c) (i), (iii) (d) (i), (ii), (iii)

21. If A is a square matrix such that $A^2 = A$, then $(I + A)^3 - 7A$ is equal to

(a) A (b) $I - A$ (c) I (d) $3A$

22. The determinant $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ is

(a) independent of θ only
(b) independent of x only
(c) independent of both θ and x
(d) none of the above

23. If $f(x)=\begin{cases}\lambda(x^2-2x), & \text{if } x\le 0\\ 4x+1, & \text{if } x>0\end{cases}$, then which one of the following is correct ?

(a) $f(x)$ is continuous at $x = 0$ for any value of λ
(b) $f(x)$ is discontinuous at $x = 0$ for any value of λ
(c) $f(x)$ is discontinuous at $x = 1$ for any value of λ
(d) None of the above

24. The point on the curve $y = (x - 2)^2$ at which the tangent is parallel to the chord joining the points (2, 0) and (4, 4), is

(a) (1, 3) (b) (3, 1) (c) (1, 0) (d) (0, 1)

25. Solution of the differential equation $ye^{\frac{x}{y}}dx = (xe^{\frac{x}{y}} + y^2)dy$ $(y \neq 0)$ is

(a) $e^{\frac{x}{y}} = y + C$ (b) $e^{\frac{x}{y}} = x + C$
(c) $e^x + y + C$ (d) $e^y = x + C$

26. Find the equation of the plane, which is perpendicular to the plane $5x + 3y + 6z + 8 = 0$ and which contains the line of intersection of the planes $x + 2y + 3z - 4 = 0$ and $2x + y - z + 5 = 0$.

(a) $51x - 15y + 50z + 173 = 0$
(b) $51x + 15y - 50z + 173 = 0$
(c) $15x + 15y + 50z + 173 = 0$
(d) none of these

27. How many times must a man toss a fair coin, so that the probability of having atleast one head is more than 80% ?

(a) Atleast 3 (b) Atleast 5
(c) Atmost 3 (d) Atmost 5

28. Maximize $Z = 10x_1 + 25x_2$, subject to $0 \le x_1 \le 3$, $0 \le x_2 \le 3$, $x_1 + x_2 \le 5$.

(a) 80 at (3, 2) (b) 75 at (0, 3)
(c) 30 at (3, 0) (d) 95 at (2, 3)

29. The value of $\lim_{x\to\alpha}\frac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2}$, where α and β are the roots of the equation $ax^2 + bx + c = 0$, is

(a) $\frac{(\alpha-\beta)^2}{2}$ (b) $(a-b)^2$

(c) $\frac{\alpha^2(a-b)^2}{2}$ (d) $\frac{a^2(\alpha-\beta)^2}{2}$

30. The mean of five numbers is 0 and their variance is 2. If three of these numbers are –1, 1 and 2, then the other two numbers are

(a) – 5 and 3 (b) – 4 and 2

(c) – 3 and 1 (d) – 2 and 0

31. If the third term in the expansion of $\left(\frac{1}{x}+x^{\log_{10}x}\right)^5$ is 1000, then x =

(a) 10^2 (b) 10^3 (c) 10 (d) 10^4

32. If the latus rectum of an ellipse with axis along x-axis and centre at origin is 10, distance between foci = length of minor axis, then the equation of the ellipse is

(a) $\frac{x^2}{25}+\frac{y^2}{10}=1$ (b) $\frac{x^2}{50}+\frac{y^2}{100}=1$

(c) $\frac{x^2}{5}+\frac{y^2}{5}=1$ (d) $\frac{x^2}{100}+\frac{y^2}{50}=1$

33. For all $n \in N$, $41^n - 14^n$ is a multiple of

(a) 26 (b) 27

(c) 25 (d) none of these

34. Number of ways in which 3 boys and 3 girls (all are of different heights) can be arranged in a line so that boys as well as girls among themselves are in decreasing order of their heights (from left to right), is

(a) 6! (b) 3! × 3! × 2!

(c) 10 (d) 20

35. If $\cos^{-1}x+\cos^{-1}y=\frac{\pi}{2}$ and $\tan^{-1}x-\tan^{-1}y=0$, then $x^2 + xy + y^2$ is equal to

(a) 0 (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{3}{2}$ (d) $\frac{1}{8}$

36. Using cofactors of elements of third row, evaluate $\Delta=\begin{vmatrix}1 & x & y+z\\ 1 & y & z+x\\ 1 & z & x+y\end{vmatrix}$.

(a) 1 (b) –1 (c) 0 (d) 2

37. Let $f(x) = \sin x$, $g(x) = x^2$ and $h(x) = \log_e x$. If $F(x) = (hogof)(x)$, then $F''(x)$ is equal to

(a) $a\ \text{cosec}^3 x$ (b) $2\cot x^2 - 4x^2\text{cosec}^2 x^2$

(c) $2x\cot x^2$ (d) $-2\ \text{cosec}^2 x$

38. Evaluate $\int\frac{\sqrt{\tan x}}{\sin x\cdot\cos x}dx$

(a) $3\sqrt{\tan x}+c$ (b) $\sqrt{\tan x}+c$

(c) $2\sqrt{\tan x}+c$ (d) $\frac{\sqrt{\tan x}}{2}+c$

39. Find the order and degree of the differential equation, if defined $y=x\frac{dy}{dx}+\sqrt{1+\left[\frac{dy}{dx}\right]^2}$.

(a) 1, 2 (b) 2, 2 (c) 1, 3 (d) 2, 3

40. Two integers are selected at random from integers 1 to 11. If the sum is even, then find the probability that both the numbers selected are odd.

(a) $\frac{1}{5}$ (b) $\frac{4}{5}$ (c) $\frac{2}{5}$ (d) $\frac{3}{5}$

41. The relation S defined on the set $N \times N$ by $(a, b)S(c, d) \Rightarrow a + d = b + c$ is an

(a) equivalence relation

(b) reflexive but not symmetric

(c) only transitive (d) only symmetric

42. If $\vec{a}=\hat{i}+\hat{j}+\hat{k}, \vec{b}=4\hat{i}-2\hat{j}+3\hat{k}$ and $\vec{c}=\hat{i}-2\hat{j}+\hat{k}$, then find a vector of magnitude 6 units which is parallel to the vector $2\vec{a}-\vec{b}+3\vec{c}$.

(a) $\pm(2\hat{i}-4\hat{j}+4\hat{k})$ (b) $\pm(\hat{i}-2\hat{j}+2\hat{k})$

(c) $\pm(\hat{i}-\hat{j}+\hat{k})$ (d) $\pm(\hat{i}+\hat{j}+\hat{k})$

43. If the trace of the matrix

$$A=\begin{bmatrix}x-1 & 0 & 2 & 5\\ 3 & x^2-2 & 4 & 1\\ -1 & -2 & x-3 & 1\\ 2 & 0 & 4 & x^2-6\end{bmatrix}$$

is 0, then x is equal to

(a) {2, 3} (b) {–2, –3}

(c) {–3, 2} (d) {1, 2}

44. If $f(x)$ defined by the following is continuous at $x = 0$, then the values of and c are

$$f(x)=\begin{cases}\dfrac{\sin(a+1)x+\sin x}{x}, & \text{if } x<0\\ c, & \text{if } x=0\\ \dfrac{\sqrt{x+bx^2}-\sqrt{x}}{bx^{\frac{3}{2}}}, & \text{if } x>0\end{cases}$$

(a) $-\frac{3}{2},\frac{1}{2}$ (b) $\frac{1}{2},\frac{1}{2}$ (c) $\frac{3}{2},1$ (d) $-\frac{1}{2},\frac{1}{2}$

45. An open box with a square base is to be made out of a given quantity of metal sheet of area c^2, then the maximum volume of the box is

(a) $\frac{c^3}{6\sqrt{3}}$ (b) $\frac{c^2}{\sqrt{3}}$ (c) $\frac{c^3}{6}$ (d) $\frac{c^3}{3\sqrt{3}}$

SOLUTIONS

1. (a) : $(fof)x = f\left(\frac{x}{x-1}\right)=\frac{\frac{x}{x-1}}{\left(\frac{x}{x-1}\right)-1}=x$

$\Rightarrow$ $(fofof)x = f(fof)x = f(x)=\frac{x}{x-1}$

$\therefore$ $(fofof.....19 \text{ times})(x)=\frac{x}{x-1}$

2. (c) : $\because$ a, b, c are in H.P. $\therefore$ $b=\frac{2ac}{a+c}$ ...(i)

b, c, d are in G.P. $\therefore$ $c^2 = bd$ ...(ii)

and c, d, e are in A.P. $\therefore$ $d=\frac{c+e}{2}$ ...(iii)

From (i), $ab + bc = 2ac$

$\Rightarrow$ $c=\frac{ab}{2a-b}$ ...(iv)

Also, $d=\frac{1}{2}\left(\frac{ab}{2a-b}+e\right)$ (Using (iii) and (iv)) ...(v)

On putting the values of c and d from (iv) and (v) in (ii), we get

$$\frac{a^2b^2}{(2a-b)^2}=\frac{b}{2}\left[\frac{ab}{2a-b}+e\right]$$

$$\Rightarrow\ e=\frac{ab^2}{(2a-b)^2}$$

3. (c) : Let $z - 1 = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$\therefore$ Given expression $= re^{i\theta}\cdot e^{-i\alpha}+\frac{1}{re^{i\theta}}\cdot e^{i\alpha}$

$$=re^{i(\theta-\alpha)}+\frac{1}{r}e^{-i(\theta-\alpha)}$$

Since, imaginary part of given expression is zero, we have

$$r\sin(\theta-\alpha)-\frac{1}{r}\sin(\theta-\alpha)=0$$

$\Rightarrow$ $r^2 - 1 = 0 \Rightarrow r = 1 \Rightarrow |z - 1| = 1$

or, $\sin(\theta - \alpha) = 0 \Rightarrow \theta - \alpha = 0 \Rightarrow \theta = \alpha$

$\Rightarrow$ $\arg(z - 1) = \alpha$

4. (a) : $(a^2 - 1)x^2 + 2(a - 1)x + 2$ is positive for all x, if

$a^2 - 1 > 0$ and $4(a - 1)^2 - 8(a^2 - 1) < 0$

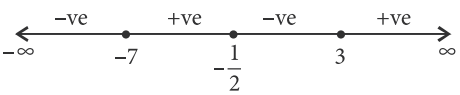
$\Rightarrow$ $a^2 - 1 > 0$ and $(a - 1)(a + 1) > 0$

$\Rightarrow$ $a < -1$ or $a > 1$

5. (a)

6. (a) : $(2x + 1)(x - 3)(x + 7) = 0$

$\Rightarrow$ $x = -\frac{1}{2}, 3, -7$



Clearly, $(2x + 1)(x - 3)(x + 7) < 0$

when $x < -7$ or $-\frac{1}{2} < x < 3$

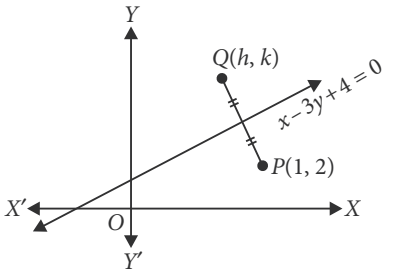
$\therefore$ $x \in (-\infty, -7) \cup \left(-\frac{1}{2}, 3\right)$.

7. (d) : Every number between 100 and 1000 is a 3-digit number. We first have to count the permutations of 6 digits taken 3 at a time. This number would be ${}^6P_3$. But, these permutations will include those also where 0 is at the 100^{th} place. To get the number of such numbers, we fix 0 at the 100^{th} place and rearrange the remaining 5 digits taking 2 at a time. This number is ${}^5P_2$.

So, the required number $= {}^6P_3 - {}^5P_2 = \frac{6!}{3!}-\frac{5!}{3!}$

$= (4 \times 5 \times 6) - (4 \times 5) = 100$

8. (c) : Let $Q(h, k)$ be the image of the point $P(1, 2)$ in the line $x - 3y + 4 = 0$ ...(i)



Hence, slope of line PQ

$$= \frac{-1}{\text{Slope of line } x-3y+4=0}$$

so that $\frac{k-2}{h-1} = \frac{-1}{\frac{1}{3}}$ or $3h + k = 5$ ...(ii)

Also, $\frac{h+1}{2} - 3\left(\frac{k+2}{2}\right) + 4 = 0$ or $h - 3k = -3$ ...(iii)

Solving (ii) and (iii), we get $h = \frac{6}{5}$ and $k = \frac{7}{5}$.

9. (b) : Given, $\frac{x^2}{14-a} + \frac{y^2}{9-a} = 1$

The equation will represent an ellipse if
$14 - a > 0$ and $9 - a > 0$
$\Rightarrow$ $a < 14$ and $a < 9 \Rightarrow a < 9$
a hyperbola if $14 - a > 0$ and $9 - a < 0$
$\Rightarrow$ $-a > -14$ and $a > 9 \Rightarrow 9 < a < 14$

10. (b) : $\lim\limits_{x\to 0} \frac{2\sin x - \sin 2x}{x^3}$

$$= \lim_{x\to 0} \frac{2\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots.\right) - \left(2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \ldots.\right)}{x^3}$$

$$= \lim_{x\to 0} \frac{\frac{-2x^3}{3!} + \frac{2x^5}{5!} - \ldots.\left(\frac{8x^3}{3!} - \frac{32x^5}{5!} + \ldots.\right)}{x^3}$$

$$= -\frac{2}{3!} + 0 - \ldots.\left(\frac{8}{3!} - 0 + \ldots.\right) = -\frac{2}{6} + \frac{8}{6} = \frac{6}{6} = 1$$

11. (a) : Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$. Then, we have

$$\vec{a} \times \hat{i} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ 1 & 0 & 0 \end{vmatrix} = a_3\hat{j} - a_2\hat{k}$$

Therefore, $|\vec{a} \times \hat{i}|^2 = a_2^2 + a_3^2$

Similarly, $|\vec{a} \times \hat{j}|^2 = a_1^2 + a_3^2$ and $|\vec{a} \times \hat{k}|^2 = a_1^2 + a_2^2$

Hence, $|\vec{a} \times \hat{i}|^2 + |\vec{a} \times \hat{j}|^2 + |\vec{a} \times \hat{k}|^2$

$$= (a_2^2 + a_3^2) + (a_1^2 + a_3^2) + (a_1^2 + a_2^2)$$

$$= 2(a_1^2 + a_2^2 + a_3^2) = 2|\vec{a}|^2$$

12. (c)

13. (c) : A and B can be arranged in ${}^{12}P_2$ $= 11 \times 12$ ways. Since we want 4 letters in between A and B.

$\therefore$ A and B can take the following places.

Place for A	Place for B
1	6
2	7
3	8
4	9
5	10
6	11
7	12

A and B can be interchanged. Therefore required probability is $\frac{14}{11\times 12} = \frac{7}{66}$

14. (b) : Since the birth date of any person can fall in anyone of the 12 months, the number of total outcomes is 12^6.

Let E : Event that the birth dates of all 6 fall in two different months.

$$P(E) = \frac{{}^{12}C_2(2^6 - 2)}{12^6} = \frac{66\times 62}{12^6} = \frac{11\times 31}{12^5} = \frac{341}{12^5}$$

15. (b) : Since $x + y + z = 1$, we have

$$\frac{4}{x} + \frac{9}{y} + \frac{16}{z} = (x+y+z)\left(\frac{4}{x} + \frac{9}{y} + \frac{16}{z}\right)$$

$$= (4+9+16) + \left(\frac{4z}{x} + \frac{16x}{z}\right) + \left(\frac{9z}{y} + \frac{16y}{z}\right) + \left(\frac{4y}{x} + \frac{9x}{y}\right)$$

$$\geq 29 + 2\sqrt{64} + 2\sqrt{144} + 2\sqrt{36} = 81$$

Equality holds if and only if $x = \frac{2}{9}, y = \frac{3}{9}, z = \frac{4}{9}$

16. (b) : $8R^2 = 4R^2(\sin^2 A + \sin^2 B + \sin^2 C)$

$$\Rightarrow \quad 2 = \frac{1-\cos 2A}{2} + \frac{1-\cos 2B}{2} + \frac{1-\cos 2C}{2}$$

$$\Rightarrow \quad -1 = \cos 2A + \cos 2B + \cos 2C$$
$$= -1 - 4\cos A \cos B \cos C$$

Hence, $\cos A \cos B \cos C = 0$

So one of the angles of the triangle is a right angle.

17. (a) : Since, $1 + \sin x \sin^2 \frac{x}{2} = 0$

$$\therefore \quad 1 + \sin x\left(\frac{1-\cos x}{2}\right) = 0$$

$$\Rightarrow \quad 2 + \sin x - \sin x \cos x = 0$$

$$\Rightarrow \quad \sin 2x - 2\sin x = 4$$

which is not possible for any x in $[-\pi, \pi]$

18. (d) : $\int_0^2 [x^2]dx = \int_0^1 [x^2]dx + \int_0^{\sqrt{2}} [x^2]dx + \int_{\sqrt{2}}^{\sqrt{3}} [x^2]dx + \int_{\sqrt{3}}^{2} [x^2]dx$

$$= \int_0^1 0\,dx + \int_1^{\sqrt{2}} 1\,dx + \int_{\sqrt{2}}^{\sqrt{3}} dx + \int_{\sqrt{3}}^{2} 3\,dx$$

$$= 0 + 1(\sqrt{2} - 1) + 2(\sqrt{3} - \sqrt{2}) + 3(2 - \sqrt{3})$$

$$= 5 - \sqrt{3} - \sqrt{2}$$

19. (d) : The given function is $f : (2, 3) \to (0, 1)$ defined by $f(x) = x - [x]$

Let $y \in (0, 1)$ be such that $y = x - 2$

$\{\because 2 < x < 3 \Rightarrow [x] = 2\}$

$$\Rightarrow \quad x = y + 2 \Rightarrow f^{-1}(x) = x + 2$$

20. (c)

21. (c) : Here, $A^2 = A$

$$\therefore \quad (I + A)^3 - 7A = I^3 + A^3 + 3I^2A + 3A^2I - 7A$$
$$= I + A^3 + 3A + 3A^2 - 7A$$
$$= I + A \cdot A - A = I + A - A = I \quad (\because A^2 = A)$$

22. (a) : Let $\Delta = \begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$

$$\Rightarrow \quad \Delta = x\begin{vmatrix} -x & 1 \\ 1 & x \end{vmatrix} - \sin\theta \begin{vmatrix} -\sin\theta & 1 \\ \cos\theta & x \end{vmatrix} + \cos\theta \begin{vmatrix} -\sin\theta & -x \\ \cos\theta & 1 \end{vmatrix}$$

$$= x(-x^2 - 1) - \sin\theta\,(-x\sin\theta - \cos\theta) + \cos\theta\,(-\sin\theta + x\cos\theta)$$
$$= -x^3 - x + x\sin^2\theta + \sin\theta\cos\theta - \sin\theta\cos\theta + x\cos^2\theta$$
$$= -x^3$$

23. (b)

24. (b) : Slope of chord joining (2, 0) and (4, 4) is

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{4-0}{4-2} = \frac{4}{2} = 2 \qquad \ldots(i)$$

Equation of given curve is $y = (x - 2)^2$

$$\Rightarrow \quad \frac{dy}{dx} = 2(x-2)$$

Now, from (i), we get

$2(x - 2) = 2 \Rightarrow x = 3 \Rightarrow y = (3 - 2)^2 = 1$

Hence, the required point is (3, 1).

25. (a)

26. (b) : Since $(x + 2y + 3z - 4) + \lambda(2x + y - z + 5) = 0$

$\Rightarrow x(1 + 2\lambda) + y(2 + \lambda) + z(3 - \lambda) - 4 + 5\lambda = 0$ ...(i)

This is perpendicular to the plane

$5x + 3y + 6z + 8 = 0$

$\therefore \quad 5(1 + 2\lambda) + 3(2 + \lambda) + 6(3 - \lambda) = 0$

$$\Rightarrow \quad 7\lambda + 29 = 0 \Rightarrow \lambda = -\frac{29}{7}$$

On putting $\lambda = -29/7$ in (i), we have the equation of the required plane as

$$x\left(1 - \frac{58}{7}\right) + y\left(2 - \frac{29}{7}\right) + z\left(3 + \frac{29}{7}\right) - 4 - \frac{145}{7} = 0$$

$$\Rightarrow \quad 51x + 15y - 50z + 173 = 0$$

27. (a) : Let man tosses the coin n times.

$P(X \geq 1) > \frac{80}{100}$, where, X is the number of heads.

$$\therefore \quad 1 - P(X = 0) > \frac{80}{100} \Rightarrow 1 - {}^nC_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^n > \frac{8}{10}$$

$$\Rightarrow \quad 1 - \frac{1}{2^n} > \frac{4}{5} \Rightarrow \frac{1}{2^n} < \frac{1}{5} \Rightarrow 2^n > 5 \qquad \ldots(i)$$

Inequality (i) is satisfied for $n \geq 3$.

Hence, coin must be tossed 3 or more times.

28. (d) **29. (d)**

30. (d) : Let the other two numbers be x and y.

According to question,

$$\frac{-1 + 1 + 2 + x + y}{5} = 0 \Rightarrow x + y = -2 \qquad \ldots(i)$$

Also, $\sigma^2 = 2$

$$\Rightarrow \frac{(-1-0)^2+(1-0)^2+(2-0)^2+(x-0)^2+(y-0)^2}{5} = 2$$

$\Rightarrow$ $1 + 1 + 4 + x^2 + y^2 = 10 \Rightarrow x^2 + y^2 = 4$ ...(ii)

$\Rightarrow$ $(x + y)^2 - 2xy = 4 \Rightarrow xy = 0$ ...(iii)

Now, $(x - y)^2 = x^2 + y^2 - 2xy = 4 - 0 = 4$

$\Rightarrow$ $x - y = \pm 2$ ...(iv)

$\Rightarrow$ $x = 0, y = -2$ or $x = -2, y = 0$ (from (i) and (iv)

31. (a) : Given, $T_3 = 1000$

$$\therefore \quad {}^5C_2\left(\frac{1}{x}\right)^3\left(x^{\log_{10}x}\right)^2 = 1000$$

$$\Rightarrow \quad 10\,x^{-3} \times x^{2\log_{10}x} = 1000$$

$$\Rightarrow \quad \left(2\log_{10}x - 3\right) = \log_x 10^2$$

$$\Rightarrow \quad (2\log_{10}x - 3) = \frac{2}{\log_{10}x}$$

$$\Rightarrow \quad 2t - 3 = \frac{2}{t}, \text{ where } t = \log_{10}x$$

$\Rightarrow$ $(2t + 1)(t - 2) = 0 \Rightarrow t = 2$ ($t \neq -1/2$, neglected)

$\therefore$ $\log_{10}x = 2 \Rightarrow x = 10^2 = 100$

32. (d)

33. (b) : Let $P(n)$ be the statement given by $P(n) : 41^n - 14^n$ is a multiple of 27.

For $n = 1$, *i.e.*, $P(1) = 41^1 - 14^1 = 27 = 1 \times 27$ which is a multiple of 27 $\therefore$ $P(1)$ is true.

Let $P(k)$ be true *i.e.*, $41^k - 14^k = 27\lambda$ ... (i)

For $n = k + 1$, $41^{k+1} - 14^{k+1}$

$= 41^k 41 - 14^k 14 = (27\lambda + 14^k)\,41 - 14^k 14$ [using (i)]

$= (27\lambda \times 41) + (14^k \times 27) = 27(41\lambda + 14^k)$

which is a multiple of 27.

Therefore, $P(k+1)$ is true when $P(k)$ is true. Hence, from the principle of mathematical induction, the statement is true for all natural numbers n.

34. (d) : Since order of boys and girls are to be maintained in any of the different arrangements.

$$\therefore \quad \text{Required number} = \frac{6!}{3!3!} = 20.$$

35. (c) **36. (c)**

37. (d) **38. (c)**

39. (a) : $$y = x\frac{dy}{dx} + \sqrt{1+\left[\frac{dy}{dx}\right]^2}$$

$$\Rightarrow \quad y - x\frac{dy}{dx} = \sqrt{1+\left[\frac{dy}{dx}\right]^2}$$

$$\Rightarrow \quad (x^2-1)\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} + y^2 - 1 = 0$$

(squaring on both sides)

which represents a quadratic polynomial in $\frac{dy}{dx}$.

$\therefore$ Order of the differential equation = 1

and degree of the differential equation = 2

40 (d) : Let E = Event of selecting both odd numbers.

F = Event that the sum of chosen number is even in integers from 1 to 11, here 5 even and 6 odd integers.

$$\therefore \quad P(E) = \frac{{}^6C_2}{{}^{11}C_2} = \frac{\left(\frac{6\times5}{2\times1}\right)}{\left(\frac{11\times10}{2\times1}\right)} = \frac{6\times5}{11\times10} = \frac{3}{11}$$

and $P(F) = P$(both numbers are even) + P(both numbers are odd)

$$= \frac{{}^6C_2}{{}^{11}C_2} + \frac{{}^5C_2}{{}^{11}C_2} = \frac{\left(\frac{6\times5}{2\times1}\right)}{\left(\frac{11\times10}{2\times1}\right)} + \frac{\left(\frac{5\times4}{2\times1}\right)}{\left(\frac{11\times10}{2\times1}\right)} = \frac{5}{11}$$

and $$P(E\cap F) = \frac{{}^6C_2}{{}^{11}C_2} = \frac{\left(\frac{6\times5}{2\times1}\right)}{\left(\frac{11\times10}{2\times1}\right)} = \frac{3}{11}$$

$$\therefore \quad \text{Required probability} = P\left(\frac{E}{F}\right) = \frac{P(E\cap F)}{P(F)} = \frac{3}{5}$$

41. (a)

42. (a) : Let the vector be $\vec{r} = \lambda(2\vec{a} - \vec{b} + 3\vec{c})$

$\Rightarrow$ $\vec{r} = \lambda(2\hat{i} + 2\hat{j} + 2\hat{k} - 4\hat{i} + 2\hat{j} - 3\hat{k} + 3\hat{i} - 6\hat{j} + 3\hat{k})$

$\Rightarrow$ $\vec{r} = \lambda(\hat{i} - 2\hat{j} + 2\hat{k})$

$\therefore$ $|\vec{r}| = \pm\lambda|\hat{i} - 2\hat{j} + 2\hat{k}|$

$\Rightarrow$ $\pm\lambda\sqrt{1+4+4} = 6 \Rightarrow \pm\lambda \cdot 3 = 6 \Rightarrow \lambda = \pm 2$

Therefore, $\vec{r} = \pm 2(\hat{i} - 2\hat{j} + 2\hat{k}) = \pm(2\hat{i} - 4\hat{j} + 4\hat{k})$

43. (c) : Trace of matrix is defined as

$$\sum_{i=1}^{n} a_{ii} = 2x^2 + 2x - 12 = 0 \Rightarrow x = -3, 2$$

44. (a) **45. (a)**

10 BEST PROBLEMS

1. Given that $iz^2 = 1 + \frac{2}{z} + \frac{3}{z^2} + \frac{4}{z^3} + \frac{5}{z^4} + ...$ and $z = n \pm \sqrt{-i}$. Find the value of n.

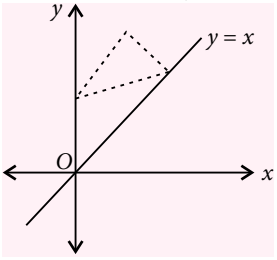
2. Find the value of x by evaluating the given series

$$1 + \frac{1}{5} + \frac{1\times3}{5\times10} + \frac{1\times3\times5}{5\times10\times15} + ... \infty = \sqrt{x},\ x \in Q$$

3. If r_1, r_2, r_3, r_4, r_5 are the complex roots of the equation $x^5 - 3x^4 - 1 = 0$. Find the value of $\frac{1}{r_1^9} + \frac{1}{r_2^9} + \frac{1}{r_3^9} + \frac{1}{r_4^9} + \frac{1}{r_5^9}$.

4. Find the sum of all (distinct) complex values of c for which the polynomial $f_c(x) = x^4 - (c^2 - 7c + 11)x^2 + (18 - 21c + 8c^2 - c^3)$ has strictly less than four distinct complex zeroes.

5. The minimum possible perimeter of a triangle with one vertex at (3, 9), one anywhere on the y-axis and one anywhere on the line $y = x$ is



6. $\lim\limits_{n\to\infty} \sum\limits_{r=3}^{n} \frac{r^3 - 8}{r^3 + 8} =$

7. Inside a square $ABCD$ points P and Q are positioned so that $DP \parallel QB$ and $DP = PQ = QB$. Of all configurations that satisfy these requirements, what is the minimum possible value of $\angle ADP$, (in degrees)?

8. Find the value of $\int_0^{2n\pi} \max\{\sin x, \sin^{-1}(\sin x)\}dx$. Draw the graph of the same.

9. Let P be a point (other than the origin) lying on the parabola $y = x^2$. The normal line to the parabola at P will intersect the parabola at another point Q. The minimum possible value for the area bounded by the line PQ and the parabola is

10. Find the value of $\lim\limits_{x\to0} \left(\int_0^1 (by + a(1-y))^x dy \right)^{1/x}$ where $b > a$.

SOLUTIONS

1. Multiplying both sides of the equation by z, we get

$$iz^3 = z + 2 + \frac{3}{z} + \frac{4}{z^2} + ...$$

and subtracting the original equation from this one, we get

$$iz^2(z-1) = z + 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + ...$$

Using the formula for an infinite geometric series, we find

$$iz^2(z-1) = \frac{z}{1 - \frac{1}{z}} = \frac{z^2}{z-1}$$

Rearranging, we get

$$iz^2(z-1)^2 = z^2 \Rightarrow (z-1)^2 = \frac{1}{i} \Rightarrow z = 1 \pm \sqrt{-i}.$$

Thus $n = 1$

2. Notice that each term is of the following form:

$$\frac{\prod\limits_{k=1}^{n}(2k-1)}{\prod\limits_{k=1}^{n}(5k)}$$

$$= \frac{\prod\limits_{k=1}^{n}(2k-1)\cdot\prod\limits_{k=1}^{n}(2k)}{(n!)5^n\prod\limits_{k=1}^{n}(2k)} = \frac{(2n)!}{(n!)5^n(n!)2^n} = \frac{1}{10^n}\binom{2n}{n}$$

Here $\binom{n}{r} = {}^nC_r$

Hence we need to find $s=\sum_{n=0}^{\infty}\frac{1}{10^n}\binom{2n}{n}$

Now since the result is the square root of a rational number, let's find s^2. Using the Cauchy Product (with 1/10 as the independent variable), we get the following formula.

$$s^2=\sum_{n=0}^{\infty}\frac{1}{10^n}\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}$$

Now it can be shown that for all whole numbers n we have

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}=4^n$$

Hence, we have

$$s^2=\sum_{n=0}^{\infty}\frac{1}{10^n}=4^n=\sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^n=\frac{1}{1-\frac{2}{5}}=\frac{5}{3}$$

Thus, $x = 5/3$

3. From the given polynomial, we have

$$\sum r_i=3 \quad \text{and} \quad \sum r_i^{-1}=0$$

The sum of the reciprocals of the roots come from the fact that the polynomial with reciprocal roots has its coefficients reversed.

$$x^5 - 3x^4 - 1 = 0 \implies x^4(x-3)=1$$

$$x^4=(x-3)^{-1} \implies x^{-9}=\frac{(x-3)^2}{x}$$

$$\sum r_i^{-9}=\sum\frac{(r_i-3)^2}{r_i}=\sum\left(\frac{r_i^2-6r_i+9}{r_i}\right)$$

$$=\sum(r_i-6+9r_i^{-1})=\sum r_i-\sum 6+9\sum r_i^{-1}$$

$$= 3 - 5(6) + 0 = -27.$$

4. The polynomial $f_c(x)$ will fail to have four distinct complex zeroes when the quadratic polynomial

$g_c(x) = x^2 - (c^2 - 7c + 11)x + (18 - 21c + 8c^2 - c^3)$ either has repeated roots or has 0 as a root.

Case1: One of the roots of $g_c(x)$ is zero precisely when

$$c^3 - 8c^2 + 21c - 18 = (c-2)(c-3)^2 = 0$$

and so precisely when $c = 2, 3$

Case 2 : The roots of $g_c(x)$ are repeated when its discriminant is zero, so that

$$(c^2 - 7c + 11)^2 - 4(18 - 21c + 8c^2 - c^3) = 0$$

$$c^4 - 10c^3 + 39c^2 - 70c + 49 = 0$$

$$(c^2 - 5c + 7)^2 = 0$$

and this happens for two distinct values of c which sum to 5.

Thus the sum of the possible values of c is $2 + 3 + 5 = 10$.

5. Let A be the vertex (3, 9), B be the vertex on the y-axis and C be the vertex on the line $y = x$. Also let D (–3, 9) be the reflection of A in the y-axis and E(9, 3) be the reflection of A in the line $y = x$. Then $AB = BD$ and $AC = CE$, and thus the perimeter of ΔABC is equal to $DB + BC + CE$.

But the shortest distance between two points is a straight line, so

$$DB + BC + CE \geq DE = \sqrt{(9-(-3))^2+(3-9)^2}$$

$$=\sqrt{180}=6\sqrt{5}$$

This minimum can be obtained by then choosing B and C as the points of intersection of the line DE with the y-axis and the line $y = x$, respectively. This gives us the points $B\left(0,\frac{15}{2}\right)$ and $C(5, 5)$. This will yield a perimeter for ΔABC of $6\sqrt{5}$.

6. $$\lim_{n\to\infty}\left(\frac{3^3-8}{3^3+8}\right)\left(\frac{4^3-8}{4^3+8}\right).....\left(\frac{n^3-8}{n^3+8}\right)$$

$$=\lim_{n\to\infty}\left(\frac{3-2}{3+2}.\frac{3^2+4+2(3)}{3^2+4-2(3)}\right)\left(\frac{4-2}{4+2}.\frac{4^2+4+2(4)}{4^2+4-2(4)}\right)....$$

$$...\left(\frac{n-2}{n+2}.\frac{n^2+4+2(n)}{n^2+4-2(n)}\right)$$

$$=\lim_{n\to\infty}\left\{\frac{3-2}{3+2}.\frac{4-2}{4+2}.\frac{5-2}{5+2}.....\frac{n-2}{n+2}\right\}$$

$$\left\{\frac{3^2+4+2(3)}{3^2+4-2(3)}.\frac{4^2+4+2(4)}{4^2+4-2(4)}.....\frac{n^2+4+2(n)}{n^2+4-2(n)}\right\}$$

$$=\lim_{n\to\infty}\left\{\frac{1\cdot2\cdot3\cdot4\cdot5\cdot6....}{5\cdot6\cdot7\cdot8....}\right\}\left\{\frac{19\cdot28\cdot39\cdot52\cdot63....}{7\cdot12\cdot19\cdot28\cdot39\cdot42\cdot63....}\right\}$$

$$=\lim_{n\to\infty}\frac{(n^2+5)(n^2+2n+4)(1\cdot2\cdot3\cdot4)}{(n-1)n(n+1)(n+2)}\left(\frac{1}{7\cdot12}\right)=\frac{2}{7}$$

7. Without loss of generality let the corners of the square be

$A(0, 2)$, $B(2, 2)$, $C(2, 0)$ and $D(0, 0)$. Now let point P have coordinates (a, b); then by symmetry the coordinates of point Q must be $(2 - a, 2 - b)$. Then since $DP = PQ$, we have that

$$a^2 + b^2 = (2 - 2a)^2 + (2 - 2b)^2$$

$\Rightarrow \quad 3a^2 + 3b^2 - 8a - 8b + 8 = 0$

$\Rightarrow \quad \left(a-\frac{4}{3}\right)^2+\left(b-\frac{4}{3}\right)^2=\frac{8}{9}$

This means that P lies on a circle centered at O $\left(\frac{4}{3},\frac{4}{3}\right)$ with radius $r=\frac{2}{3}\sqrt{2}$.

Therefore, $\angle ADP$ will be minimized when DP is tangent to this circle.

Now by symmetry OD makes an angle of 45° with the x-axis and has length $\frac{4}{3}\sqrt{2}$.

Thus $\angle PDO = 45° - \angle ADP$, and so

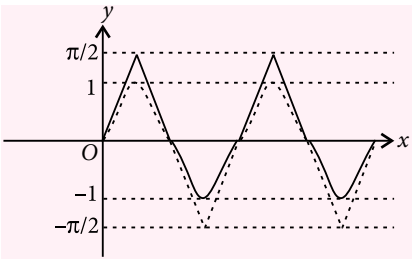
$\sin(\angle PDO) = \sin(45° - \angle ADO)$

$$= \frac{OP}{OD}=\frac{\frac{2}{3}\sqrt{2}}{\frac{4}{3}\sqrt{2}}=\frac{1}{2}$$

This implies that $45° - \angle ADP = \sin^{-1}(1/2) = 30°$, and so the minimum value for $\angle ADP$ is

$45° - 30° = 15°$

8.



The integral can be divided as

$$= n[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi}(\pi - x)dx + \int_{\pi}^{2\pi}\sin x\,dx]$$

$$= n\left[\frac{\pi^2}{8}+\frac{\pi^2}{2}-\frac{1}{2}\times\left(\pi-\frac{\pi^2}{4}\right)-2\right]$$

Solving we get, $I = n\times\frac{\pi^2-8}{4}$

9. We take a point $P_1 = (x_0, x_0^2)$ on the parabola.

Then slope of tangent is $= 2x_0$

Hence slope of normal is $\frac{-1}{2x_0}$

So equation of normal is $(x - x_0) = -2x_0(y - x_0^2)$

Solving it with the parabola we get

$$x = x_0, \left(-\frac{1}{2x_0}-x_0\right)$$

So the other point is

$$P_2=\left(-\frac{1}{2x_0}-x_0,\left(-\frac{1}{2x_0}-x_0\right)^2\right)$$

Hence the area bounded by P_1P_2 and the parabola is

$$\int_{\left(-\frac{1}{2x_0}+x_0\right)}^{x_0}\left(x_0^2+\frac{1}{2}-x-x^2\right)dx$$

(using the equation of normal)

Evaluating it we get $A=\frac{4}{3}\left(x_0+\frac{1}{4x_0}\right)^3$

By applying AM-GM inequality we get

$$\frac{x_0+\frac{1}{4x_0}}{2}\geq\sqrt{\frac{1}{4}} \quad \Rightarrow \quad \left(x_0+\frac{1}{4x_0}\right)^3\geq 1$$

Finally $A_{\min}=\frac{4}{3}$

10. $L=\lim_{x\to 0}(\int_0^1(by+a(1-y))^x dy)^{1/x}$

$I=\int_0^1(by+a(1-y))^x dy$

Let $by + a(1 - y) = t$, $(b - a)dy = dt$

$$I=\int\frac{t^x}{(b-a)}dt=\frac{t^{x+1}}{(x+1)(b-a)}+C$$

$I=\left[\frac{(by+a(1-y))^{x+1}}{(x+1)(b-a)}\right]$ computed at $y = 1$ and $y = 0$

$$I=\frac{b^{x+1}-a^{x+1}}{(x+1)(b-a)}, L=\lim_{x\to 0}\left(\frac{b^{x+1}-a^{x+1}}{(x+1)(b-a)}\right)^{1/x}$$

$$\ln(L)=\lim_{x\to 0}\frac{\ln\left(\frac{(b^{x+1}-a^{x+1})}{(x+1)(b-a)}\right)}{x}$$

As x approaches 0, denominator and numerator approaches 0.

Hence, we can use L-hospital's rule

$$\ln(L)=\lim_{x\to 0}\left(\frac{b^{x+1}\ln(b)a^{x+1}\ln(a)}{(x+1)(b-a)}-\frac{1}{x+1}\right)_1$$

$$\ln(L)=\frac{1}{b-a}\ln\left(\frac{b^b}{a^a}\right)-1 \Rightarrow L=\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{(b-a)}}$$

MOCK TEST PAPER 2015

JEE Advanced

Exam on 24th May

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SECTION-I

SINGLE CORRECT ANSWER TYPE

This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.

1. The sum of all positive integral values of a, $a \in [1, 500]$ for which the equation $[x]^3 + x - a = 0$ has solution is ([.] denote G.I.F)

(a) 462 (b) 512 (c) 784 (d) 812

2. Let $P(x)$ be a polynomial with degree 2009 and leading co-efficient unity such that $P(0) = 2008$, $P(1) = 2007$, $P(2) = 2006$, …, $P(2008) = 0$ and the value of $P(2009) = (n!) - a$ where n and a are natural number, then value of $(n + a)$ is _____

(a) 2010 (b) 2009 (c) 2011 (d) 2008

3. If circumcentre of an equilateral triangle inscribed in $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with vertices having eccentric angles α, β, γ respectively is (x_1, y_1), then $\sum \cos\alpha \cdot \cos\beta + \sum \sin\alpha \cdot \sin\beta$ is

(a) $\frac{9x_1^2}{2a^2}+\frac{9y_1^2}{2b^2}-\frac{3}{2}$ (b) $\frac{x_1^2}{2a^2}+\frac{y_1^2}{2b^2}-\frac{5}{2}$

(c) $\frac{x_1^2}{9a^2}+\frac{y_1^2}{9b^2}-\frac{5}{9}$ (d) $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-\frac{1}{2}$

4. Let $f : [0, \infty) \to R$ be a continuous and strictly increasing function such that $f^3(x)=\int_0^x t \cdot f^2(t)dt, \forall\, x>0$. The area enclosed by $y = f(x)$, the x-axis and the ordinate at $x = 3$, is

(a) 1 (b) 3/2 (c) 2 (d) 3

5. Consider the function $f(x)=\begin{cases}\frac{p(x)}{x-2}; & x\neq 2\\ 7\ ; & x=2\end{cases}$

where $P(x)$ is a polynomial such that $P'''(x)$ is identically equal to 0 and $p(3) = 9$. If $f(x)$ is continuous at $x = 2$, then $p(x)$ is

(a) $2x^2 + x + 6$ (b) $2x^2 - x - 6$

(c) $x^2 + 3$ (d) $x^2 - x + 7$

6. A particle starts to travel from a point P on the curve $C_1 : |z - 3 - 4i| = 5$, where $|z|$ is maximum. From P, the particle moves through an angle $\tan^{-1}(3/4)$ in anticlock wise direction on $|z - 3 - 4i| = 5$ and reaches at point Q. From Q, it comes down parallel to imaginary axis by 2 units and reaches at point R. Complex number corresponding to point R in the Argand plane is

(a) $(3 + 5i)$ (b) $(3 + 7i)$

(c) $(3 + 8i)$ (d) $(3 + 9i)$

7. If $\lfloor 15 = 2^{P_1}3^{P_2}5^{P_3}7^{P_4}11^{P_5}13^{P_6}$, then $\sum_{r=1}^{6} P_r$ is

(a) 24 (b) 23 (c) 22 (d) 21

8. Let $g(x)=\frac{1}{4}f(2x^2-1)+\frac{1}{2}f(1-x^2)\ \forall\, x\in R$,

where $f''(x) > 0\ \forall\ x \in R$, $g(x)$ is necessarily increasing in the interval

(a) $\left(-\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}}\right)$

(b) $\left(-\sqrt{\frac{2}{3}},0\right)\cup\left(\sqrt{\frac{2}{3}},\infty\right)$

(c) $(-1, 1)$ (d) None of these

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9. Let $A = \{x_1, x_2, x_3, ..., x_7\}$, $B = \{y_1, y_2, y_3\}$. The total number of functions $f : A \to B$ that are onto and there are exactly three elements x in A such that $f(x) = y_2$, is equal to

(a) $14 \cdot {}^7C_2$ (b) $14 \cdot {}^7C_3$
(c) $7 \cdot {}^7C_2$ (d) $7 \cdot {}^7C_3$

10. The value of $\int_3^6 (\sqrt{x+\sqrt{12x-36}} + \sqrt{x-\sqrt{12x-36}})dx$ is equal to

(a) $6\sqrt{3}$ (b) $4\sqrt{3}$ (c) $12\sqrt{3}$ (d) $2\sqrt{3}$

11. The number of the positive integer pairs (x, y) such that $\frac{1}{x}+\frac{1}{y}=\frac{1}{2007}$ where $x < y$ is

(a) 5 (b) 6
(c) 7 (d) 8

12. Let $f(x) = x^2 + \lambda x + \mu\cos x$, λ being an integer and μ a real number. The number of ordered pairs (λ, μ) for which the equations $f(x) = 0$ and $f(f(x)) = 0$ have the same (non empty) set of real roots is

(a) 4 (b) 6
(c) 8 (d) infinite

13. $a > 0$ $(a \neq 1)$, $b > 0$ $(b \neq 1)$ such that $a^{(\log_a b)^x} = b^{(\log_b a)^x}$, then $x =$

(a) 1 (b) -1 (c) 1/2 (d) 2

14. If the range of $f(x) = \begin{cases} 2+\sqrt[3]{x}, & -3 \le x < -1 \\ x^{2/3}, & -1 \le x \le 2 \end{cases}$ is $[0, \sqrt[3]{n}]$ where $n \in N$, then $n =$

(a) 1 (b) 2 (c) 4 (d) 6

15. A variable straight line with slope $m(m \neq 0)$ intersects the hyperbola $xy = 1$ at two distinct points. Then the locus of the point which divides the line segment between these two points in the ratio 1 : 2 is

(a) an ellipse (b) a hyperbola
(c) a circle (d) a parabola

16. If $\int (2 - 3\sin^2 x)\sqrt{\sec x}\, dx = 2f(x)\sqrt{g(x)} + c$ and $f(x)$ is non constant function, then

(a) $f^2(x) + g^2(x) = 1$ (b) $f^2(x) - g^2(x) = 1$
(c) $f(x)\, g(x) = 1$ (d) $f(x) = g(x)$

17. The number of ordered pairs of positive integers (a, b) such that L.C.M of a and b is $2^3 5^7 11^{13}$ is

(a) 2385 (b) 2835 (c) 3825 (d) 8325

18. $\sum_{K=1}^{10} \frac{(-1)^{K-1}}{K} \cdot ({}^{10}C_K) =$

(a) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+.....+\frac{1}{11}$

(b) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+.....+\frac{1}{10}$

(c) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+.....+\frac{1}{9}$

(d) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+.....+\frac{1}{12}$

19. The numbers a, b are chosen from the set $\{1, 2, 3, 4, ..., 10\}$ such that $a \le b$ with replacement. The probability that a divides b is

(a) $\frac{5}{11}$ (b) $\frac{29}{55}$
(c) $\frac{27}{55}$ (d) None of these

20. The area of the loop of the curve $y^2 = x^4(x + 2)$ is [in square units]

(a) $\frac{32\sqrt{2}}{105}$ (b) $\frac{64\sqrt{2}}{105}$
(c) $\frac{128\sqrt{2}}{105}$ (d) $\frac{256\sqrt{2}}{105}$

21. The sum $\sum_{k=1}^{\infty} \frac{6^k}{(3^k - 2^k)(3^{k+1} - 2^{k+1})} =$

(a) 1 (b) 2 (c) 3 (d) 4

22. The coefficient of x^6 in the expansion of $\left(1+\frac{x}{\lfloor 1}+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\frac{x^4}{\lfloor 4}+\frac{x^5}{\lfloor 5}\right)^2$ is

(a) $\frac{2}{15}$ (b) $\frac{4}{15}$ (c) $\frac{31}{360}$ (d) $\frac{2}{45}$

23. $\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}\, dx =$

(a) $\frac{\pi^2}{4}$ (b) π^2 (c) 0 (d) $\frac{\pi}{2}$

24. If a, b, c are in A.P., b, c, d are in G.P. and c, d, e are in H.P., then a, c, e are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

25. If a_1 is the greatest value of $f(x)$; where $f(x)=\left(\frac{1}{2+[\sin x]}\right)$ (where [.] denotes greatest integer function) and $a_{n+1}=\frac{(-1)^{n+2}}{(n+1)}+a_n$, then $\lim\limits_{n\to\infty}(a_n)$ is

(a) 1 (b) e^2
(c) ln2 (d) ln3

26. If $f(x)=\int\limits_1^x \frac{dt}{2+t^4}$, then

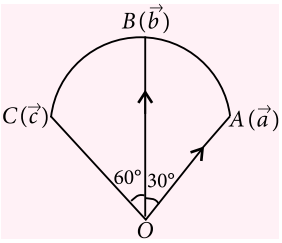
(a) $f(2)<\frac{1}{3}$ (b) $f(2)>\frac{1}{3}$
(c) $f(2)=\frac{1}{3}$ (d) $f(2) > 1$

27. Area of the region in which point $P(x, y)$, $(x > 0)$ lies; such that $y\le\sqrt{16-x^2}$ and $\left|\tan^{-1}\left(\frac{y}{x}\right)\right|\le\frac{\pi}{3}$ is

(a) $\left(\frac{16}{3}\pi\right)$ (b) $\left(\frac{8\pi}{3}+8\sqrt{3}\right)$
(c) $(4\sqrt{3}-\pi)$ (d) $(\sqrt{3}-\pi)$

28. If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of points on the circumference of a circle having centre at origin. Considering the following figure, $\vec{a}$ is equal to

(a) $(\vec{c}-2\vec{b})$
(b) $\frac{-1}{\sqrt{3}}(\vec{c}-2\vec{b})$
(c) $\frac{1}{\sqrt{3}}(\vec{c}-2\vec{b})$
(d) None of these



29. Let $A = \{1^2, 3^2, 5^2, ...\}$. If 9 elements are selected from set A to make a 3×3 matrix, then $\det(A)$ will be divisible by

(a) 9 (b) 36 (c) 8 (d) 64

30. In ΔABC, P is any point inside a triangle such that area of ΔBPC, ΔAPC, ΔAPB are equal. Line AP cut BC at M, area of ΔPMC is 5 sq. unit then area of ΔABC is

(a) 20 sq.units (b) 25 sq.units
(c) 30 sq.units (d) 10 sq.units

SECTION-II

(MULTIPLE CORRECT ANSWER TYPE)

This section contains multiple correct answer(s) type questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONE OR MORE is/are correct.

31. A hyperbola does not have mutually perpendicular tangents, then its eccentricity can be

(a) 7/6 (b) 7/5 (c) 3/2 (d) 2

32. If $\alpha, \beta\in\left(-\frac{\pi}{2}, 0\right)$ such that

$(\sin\alpha+\sin\beta)+\frac{\sin\alpha}{\sin\beta}=0$, $(\sin\alpha+\sin\beta)\frac{\sin\alpha}{\sin\beta}=-1$

and $\lambda=\lim\limits_{n\to\infty}\frac{1+(2\sin\alpha)^{2n}}{(2\sin\beta)^{2n}}$, then

(a) $\alpha=-\frac{\pi}{6}$ (b) $\lambda = 2$
(c) $\alpha=-\frac{\pi}{3}$ (d) $\lambda = 1$

33. If $f(x)=\lim\limits_{p\to\infty}\frac{x^p g(x)+h(x)+7}{7x^p+3x+1}; x\ne1$ and $f(1) = 7, f(x), g(x)$ and $h(x)$ are all continuous function at $x = 1$. Then which of the following statement(s) is/are correct?

(a) $g(1) + h(1) = 70$
(b) $g(1) - h(1) = 28$
(c) $g(1) + h(1) = 60$
(d) $g(1) - h(1) = -28$

34. Consider the ellipse $\frac{x^2}{\tan^2\alpha}+\frac{y^2}{\sec^2\alpha}=1$, where $\alpha \in (0, \pi/2)$. Which of the following quantities would vary as α varies ?

(a) Eccentricity
(b) Ordinate of the vertex
(c) Ordinates of the foci
(d) Length of the latus rectum

35. Let $R = \{(x, y) : x, y \in R, x^2 + y^2 \le 25\}$ and $R' = \left\{(x, y) : x, y \in R, y \ge \frac{4}{9}x^2\right\}$, then

(a) domain of $R \cap R' = [-3, 3]$

(b) range of $R \cap R' = [0, 4]$

(c) range of $R \cap R' = [0, 5]$

(d) $R \cap R'$ defines a function

36. The function $f(x) = \begin{cases} |x-3| & , \quad x \ge 1 \\ \left(\frac{x^2}{4}\right) - \left(\frac{3x}{2}\right) + \left(\frac{13}{4}\right), & x < 1 \end{cases}$ is

(a) continuous at $x = 1$

(b) differentiable at $x = 1$

(c) continuous at $x = 3$

(d) differentiable at $x = 3$

37. If $\lim\limits_{x\to\infty} x\left(\tan^{-1}\left(\frac{x+\lambda}{x+\mu}\right) - \frac{\pi}{4}\right) = 1$, then ordered pair(s) (λ, μ) can be

(a) (2000, 2011) (b) (0, 1)

(c) (5, 3) (d) (1, 0)

38. The number of isosceles triangles with integer sides if no side exceeds 2008 is

(a) $(1004)^2$ if equal sides do not exceed 1004

(b) $2(1004)^2$ if equal sides exceed 1004

(c) $3(1004)^2$ if equal sides have any length ≤ 2008

(d) $(2008)^2$ if equal sides have any length ≤ 2008

39. A circle of radius 4 cm is inscribed in ΔABC which touches the side BC at D. If $BD = 6$ cm, $DC = 8$ cm, then

(a) the triangle is necessarily acute angled Δ

(b) $\tan\frac{A}{2} = \frac{4}{7}$

(c) perimeter of the triangle ABC is 42 cm

(d) Area of ΔABC is 84 cm^2

40. Suppose three real numbers a, b, c are in G.P. Let $z = \frac{a+ib}{c-ib}$, then $z =$

(a) $\frac{ib}{c}$ (b) $\frac{ia}{b}$ (c) $\frac{ia}{c}$ (d) 0

41. $f : [0, 1] \to R$ is a differentiable function such that $f(0) = 0$ and $|f'(x)| \le k|f(x)|$ for all $x \in [0, 1]$, $(k > 0)$, then which of the following is/are always true ?

(a) $f(x) = 0, \forall\, x \in R$ (b) $f(x) = 0, \forall x \in [0, 1]$

(c) $f(x) \ne 0, \forall x \in [0, 1]$ (d) $f(1) = k$

42. If $\log_2(\log_{1/2}(\log_2(x))) = \log_3(\log_{1/3}(\log_3(y))) = \log_5(\log_{1/5}(\log_5(z))) = 0$ for positive x, y and z, then which of the following is/are NOT true?

(a) $z < x < y$ (b) $x < y < z$

(c) $y < z < x$ (d) $z < y < x$

43. $T_r = \frac{1}{r\sqrt{r+1} + (r+1)\sqrt{r}}$, then (here $r \in N$)

(a) $T_r > T_{r+1}$ (b) $T_r < T_{r+1}$

(c) $\sum\limits_{r=1}^{99} T_r = \frac{9}{10}$ (d) $\sum\limits_{r=1}^{n} T_r < 1$

44. If $S_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}, (n \in N)$, then $S_1 + S_2 + ... + S_{n-1}$ is equal to

(a) $nS_n - n$ (b) $nS_n - 1$

(c) $(n-1)S_{n-1} - n$ (d) $nS_{n-1} - n + 1$

45. If E and F are two independent events, such that $P(E \cap F) = \frac{1}{6}, P(E^c \cap F^c) = \frac{1}{3}$ and $(P(E) - P(F))(1 - P(F)) > 0$, then

(a) $P(E) = \frac{1}{2}$ (b) $P(E) = \frac{1}{4}$

(c) $P(F) = \frac{1}{3}$ (d) $P(F) = \frac{2}{3}$

SECTION-III

COMPREHENSION TYPE

This section contains paragraphs. Based upon each paragraph, multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.

Paragraph for Question Nos. 46 to 48

Define a function $\phi : N \to N$ as follows : $\phi(1) = 1$, $\phi(P^n) = P^{n-1}(P - 1)$, if p is prime and $n \in N$. $\phi(mn) = \phi(m)\,\phi(n)$ if m and n are relatively prime natural numbers.

46. $\phi(8n + 4)$ where $n \in N$ is equal to

(a) $\phi(4n + 2)$ (b) $\phi(2n + 1)$

(c) $2\phi(2n + 1)$ (d) $4\phi(2n + 1)$

47. The number of natural numbers 'n' such that $\phi(n)$ is odd is

(a) 1 (b) 2
(c) 3 (d) 4

48. If $\phi(7^n) = 2058$ where $n \in N$, then the value of n is

(a) 3 (b) 4
(c) 5 (d) 6

Paragraph for Question Nos. 49 to 51

If A is 3×3 matrix, then a non-trivial solution $X = (x\ y\ z)^T$ such that $AX = \lambda X (\lambda \in R)$ yields 3 values of λ say $\lambda_1, \lambda_2, \lambda_3$. For any such matrix A, λ's are called eigen values and corresponding X's are called eigen vectors. It is known that, for any 3×3 matrix, $\text{Tr}(A) = \lambda_1 + \lambda_2 + \lambda_3$, $\det A = \lambda_1\ \lambda_2\ \lambda_3$. Answer the following questions for matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

49. $\text{Tr}(A^{-1}) =$

(a) 1/3 (b) 1/2
(c) –1/2 (d) –1/3

50. $\text{Tr}(A^3) =$

(a) 149 (b) 101
(c) 128 (d) 133

51. Which of the following is false?

(a) $\exists$ a non-trivial solution X such that $AX = (2+\sqrt{7})X$

(b) $\exists$ a non-trivial solution X such that $AX = X$

(c) $\exists$ a non-trivial solution X such that $A^{-1}X = (2-\sqrt{7})X$

(d) The total number of non-trivial solutions X such that $AX = \lambda X$ is 3.

Paragraph for Question Nos. 52 to 54

Starting at (0, 0), an object moves in xy-plane via a sequence of steps, each of length 1 unit. Each step is left, right, up or down, all the four being equally likely. The probability that object reaches (2, 2) in

52. Exactly 4 steps is

(a) $\frac{5}{128}$ (b) $\frac{3}{128}$
(c) $\frac{1}{128}$ (d) $\frac{1}{256}$

53. Exactly 6 steps is

(a) $\frac{6}{4^4}$ (b) $\frac{1}{4^6}$
(c) $\frac{6}{4^6}$ (d) $\frac{15}{4^4}$

54. Six or fewer steps is

(a) $\frac{1}{16}$ (b) $\frac{1}{32}$
(c) $\frac{3}{64}$ (d) $\frac{5}{64}$

Paragraph for Question Nos. 55 to 57

Let m, n be two positive real numbers and define

$$f(n) = \int_0^\infty x^{n-1} e^{-x}\,dx \text{ and}$$

$$g(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx\,.$$

It is known that $f(n)$, for $n > 0$ is finite and $g(m, n) = g(n, m)$ for $m, n > 0$. Then,

55. $\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx =$

(a) $g(m, n)$ (b) $g(m - 1, n)$
(c) $g(m - 1, n - 1)$ (d) $g(m, n - 1)$

56. $\int_0^1 x^m \left(\log_e \frac{1}{x}\right)^n dx =$

(a) $\frac{f(n+1)}{(m+1)^n}$ (b) $\frac{f(n)}{(m+1)^{n+1}}$
(c) $\frac{f(n+1)}{(m+1)^{n+1}}$ (d) $g(m+1, n+1)$

57. $\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$

(a) $g(n, m)$ (b) $g(m - 1, n + 1)$
(c) $g(m - 1, n - 1)$ (d) $g(m + 1, n - 1)$

Paragraph for Questions Nos. 58 to 60

If $(1+x+x^2)^{100} = \sum_{r=0}^{200} a_r x^r$

58. Which of the following is true?

(a) $a_{28} = a_{72}$ (b) $a_{56} = a_{144}$
(c) $a_{200} = a_{300}$ (d) none of these

59. $a_0 + a_1 + a_2 + ... + a_{99}$ is equal to

(a) $\frac{3^{99} - a_{99}}{2}$ (b) $\frac{3^{101} - a_{99}}{2}$

(c) $\frac{3^{100} - a_{100}}{2}$ (d) none of these

60. $37a_{37}$ is equal to

(a) $64a_{36} + 165a_{35}$ (b) $64a_{35} + 148a_{36}$

(c) $56a_{32} + 168a_{22}$ (d) none of these

SECTION IV

MATRIX-MATCH TYPE

This section contains questions. Each question contains statements given in two columns which have to be matched. Statements in Column I are labeled as A, B, C and D whereas statements in Column II are labeled as p, q, r and s.

	p	q	r	s
A	(p)	(q)	(r)	(s)
B	(p)	(q)	(r)	(s)
C	(p)	(q)	(r)	(s)
D	(p)	(q)	(r)	(s)

The answers to these questions have to be appropriately bubbled as illustrate din the following examples. If the correct matches are A – p, s ; B – q, r ; C – p, q ; D – s, then the correctly bubbled matrix will look like the following:

61. Match the following:-

Column I		Column II	
(A)	Interval in which at least one of the points of local maximum of $\cos x + \sin^2 x$ lies, is	(p)	$\left(0, \frac{\pi}{2}\right)$
(B)	Interval in which $\tan^{-1}(\sin x - \cos x)$ is strictly increasing, is	(q)	$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
(C)	Interval containing the value of the integral $\int_2^3 [x^2 - 5x + 6]dx$, where $[x]$ is the greatest integer function, is	(r)	$\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$
(D)	Interval containing the value of $\int_{-3}^{3} \frac{dx}{3 + f(x)}$ if $f(x) f(-x) = 9$ for all real x, is	(s)	$\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$
		(t)	$(-\pi, \pi)$

62. Match the definite integrals in Column–I with their values in Column–II.

Column I		Column II	
(A)	$\int_0^{\pi} x(\sin^2(\sin x) + \cos^2(\cos x))dx$	(p)	$\frac{\pi^2}{16}$
(B)	$\int_0^{\pi^2/4} (2\sin\sqrt{x} + \sqrt{x}\cos\sqrt{x})dx$	(q)	$\frac{\pi^2}{2}$
(C)	$\left\lvert \frac{\pi}{\ln 2} \int_{-\pi/4}^{\pi/4} \ln(\sqrt{1+\sin 2x})dx \right\rvert$	(r)	$\frac{\pi^2}{4}$
(D)	$\int_0^{\pi} \frac{8x^3 \cos^4 x \sin^2 x}{\pi^2 - 3\pi x + 3x^2} dx$	(s)	$\frac{\pi^2}{8}$
		(t)	$\frac{\pi^2}{32}$

63. Match the following:

Column I		Column II	
(A)	Number of ways to select n objects from $3n$ objects of which n are identical and rest are different is $k^{2k-1} + \frac{1}{k}\frac{(kn)!}{(n!)^2}$, where k is	(p)	3
(B)	Number of interior point when diagonals of a convex polygon of n side intersect if no three diagonal pass through the same interior point is ${}^nC_{\lambda}$, then λ is	(q)	2
(C)	Five digit number of different digit can be made in which digit are in descending order is ${}^{10}C_{\mu}$, then μ is	(r)	4
(D)	Number of term in expansion of $(1 + 3^{1/3})^6$ which are free from radical sign is	(s)	5
		(t)	1

64. Match the following:

Column I		Column II	
(A)	Number of triangles to which an acute angle triangle ABC can act as a pedal triangle is	(p)	3
(B)	DEF is a pedal triangle of ABC. If R_1 and R_2 are circumradius of ΔDEF and ΔABC respectively, then $\frac{R_2}{R_1}$ is	(q)	2
(C)	Three points D, E, F are taken on side BC, CA and AB such that AD, BE and CF are concurrent, then $\frac{BD \cdot CE \cdot AF}{DC \cdot AE \cdot FB}$ is equal to	(r)	4
(D)	$\frac{3}{2}\left(1-\frac{r_1}{r_2}\right)\left(1-\frac{r_1}{r_3}\right)$ is (where r_1, r_2, r_3 are radii of excircles of ΔABC which is right angled at A)	(s)	1
		(t)	5

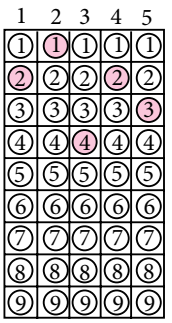
65. Match the following:

Column I		Column II	
(A)	Number of mutually perpendicular tangents that can be drawn from the curve $y = \|\|1 - e^x\| - 2\|$ to the parabola $x^2 = -4y$	(p)	2
(B)	Locus of vertex of parabola whose focus is (1, 2) and latus rectum is of 12 units is $(x - 1)^2 + (y - 2)^2 = a^2$, then $a =$	(q)	4
(C)	A line drawn through the focus F and parallel to tangent at $P(1, 2\sqrt{2})$ on the parabola $y^2 = 8x$ cut the line $y = 2\sqrt{2}$ at Q, then PQ is equal to	(r)	3
(D)	A movable parabola touches the x and y-axes at (1, 0) and (0, 1) then radius of locus of focus of parabola is	(s)	0
		(t)	5

SECTION-V

INTEGER ANSWER TYPE

This section contains questions. The answer to each of the questions is a single digit integer, ranging from 0 to 9. The appropriate bubbles below the respective question numbers in the ORS have to be darkened. For example, if the correct answers to question numbers 1, 2, 3, 4 and 5 (say) are 2, 1, 4, 2 and 3, respectively then the correct darkening of bubbles will look like as given.



66. The minimum area bounded by the function $y = f(x)$ and $y = \alpha x + 9$ ($\alpha \in R$) where f satisfies the relation $f(x+y) = f(x) + f(y) + y\sqrt{f(x)}$ $\forall\ x, y \in R$ and $f'(0) = 0$ is $9A$, value of A is

67. Let $X = \{1, 2, 3, ..., 100\}$ and Y be a subset of X such that the sum of no two elements in Y is divisible by 7. If the maximum possible number of element in Y is $40 + \lambda$, then λ is

68. Let x be in radians with $0 < x < \frac{\pi}{2}$. If $\sin(2\sin x) = \cos(2\cos x)$; then $\tan x + \cot x$ can be written as $\frac{a}{\pi^c - b}$ where $a, b, c \in N$. Then the value of $\left(\frac{a+b+c}{25}\right)$ is

69. Two lines $zi - \bar{z}i + 2 = 0$ and $z(1+i) + \bar{z}(1-i) + 2 = 0$ intersect at a point P. There is a complex number $\alpha = x + iy$ at a distance of 2 units from the point P which lies on line $z(1+i) + \bar{z}(1-i) + 2 = 0$. Find $[|x|]$ (where $[.]$ represents greatest integer function).

70. Let P, Q be two points on the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ whose eccentric angles differ by a right angle. Tangents are drawn at P and Q to meet at R. If the chord PQ divides the joint of C and R in the ratio $m : n$ (C being centre of ellipse), then find $m + n$($m : n$ is in simplified form).

71. Find the value of $\dfrac{29\int_0^1 (1-x^4)^7\,dx}{4\int_0^1 (1-x^4)^6\,dx}$.

72. If $f(x)=a\cos(\pi x)+b, f'\left(\dfrac{1}{2}\right)=\pi$ and $\int_{1/2}^{3/2} f(x)dx=\dfrac{2}{\pi}+1$, then find the value of $-\dfrac{12}{\pi}(\sin^{-1}a+\cos^{-1}b)$.

73. A sequence is obtained by deleting all perfect squares from set of natural numbers. The remainder when the 2003rd term of new sequence is divided by 2048, is

74. Let $p(x) = x^5 + x^2 + 1$ have roots x_1, x_2, x_3, x_4 and x_5, $g(x) = x^2 - 2$, then the value of $g(x_1)g(x_2)g(x_3)g(x_4)g(x_5) - 30g(x_1x_2x_3x_4x_5)$, is

75. If $\alpha = e^{i2\pi/7}$ and $f(x)=A_0+\sum_{k=1}^{20} A_kx^k$ and the value of $f(x) + f(\alpha x) + f(\alpha^2x) + ... + f(\alpha^6x)$ is $k(A_0 + A_7x^7 + A_{14}x^{14})$, then find the value of k.

SOLUTIONS

1. (d) : a is integer then x must be integer,
i.e., $[x] = x \Rightarrow a = x^3 + x$
$1 \le a \le 500 \Rightarrow 1 \le x \le 7, x \in I$

$$\sum a_i=\sum_{x=1}^{7}(x^3+x)=\left(\frac{7\cdot8}{2}\right)^2+\left(\frac{7\cdot8}{2}\right)=812$$

2. (a) : $P(x) - 2008 + x = x(x - 1)(x - 2)(x - 3)... ... (x - 2008)$
Put $x = 2009 \Rightarrow P(2009) + 1 = 2009!$

3. (a) : $A(a\cos\alpha, b\sin\alpha)$, $B(a\cos\beta, b\sin\beta)$, $C(a\cos\gamma, b\sin\gamma)$
Centroid = circumcentre = (x_1, y_1)

$$=\left(\frac{\sum a\cos\alpha}{3},\frac{\sum b\sin\alpha}{3}\right)$$

$$\Rightarrow \quad \frac{3x_1}{a}=\sum\cos\alpha,\frac{3y_1}{b}=\sum\sin\alpha$$

$$\Rightarrow \quad \left(\frac{9x_1^2}{a^2}+\frac{9y_1^2}{b^2}-3\right)=2(\sum\cos\alpha\cos\beta+\sum\sin\alpha\sin\beta)$$

$$\Rightarrow \quad \frac{9x^2}{2a^2}+\frac{9y^2}{2b^2}-\frac{3}{2}=\Sigma\cos\alpha\cos\beta+\Sigma\sin\alpha\sin\beta$$

4. (b) : $f'(x)=\dfrac{x}{3} \Rightarrow f(x)=\dfrac{x^2}{6}; A=\dfrac{1}{6}\int_0^3 x^2dx=3/2$

5. (b) : Since $p'''(x) = 0$
Let $p(x) = ax^2 + bx + c$, $p(2) = 0$
$\Rightarrow \quad 4a + 2b + c = 0 \quad ... (1)$
$p(3) = 9$
$\Rightarrow \quad 9a + 3b + c = 9 \quad ... (2)$
$p'(2) = 7 \Rightarrow 4a + b = 7 \quad ... (3)$
Solving (1), (2) and (3), we get a, b and c

6. (b) : $|z - 3 - 4i| = 5$
$\Rightarrow (x - 3)^2 + (y - 4)^2 = 25 \Rightarrow R$ is (3, 7)

7. (a) : $15! = 2^{11} \times 3^6 \times 5^3 \times 7^2 \times 11^1 \times 13^1$

$$\therefore \quad \sum_{r=1}^{6} P_r = 11+6+3+2+1+1=24$$

8. (b) : $f''(x) > 0$
$\Rightarrow$ $f'(x)$ is increasing function.
To find where g is necessarily increasing :
g is increasing $\Rightarrow g' > 0$

$$\Rightarrow \quad \frac{1}{4}\cdot f'(2x^2-1)(4x)+\frac{1}{2}f'(1-x^2)(-2x)>0$$

$$\Rightarrow \quad x\{f'(2x^2-1)-f'(1-x^2)\}>0$$

Case 1 : For $x > 0$ ... (1) $f'(2x^2 - 1) > f'(1 - x^2)$
$\Rightarrow \quad 2x^2 - 1 > 1 - x^2$

$$\Rightarrow \quad x\in\left(-\infty,\sqrt{\frac{2}{3}}\right)\cup\left(\sqrt{\frac{2}{3}},\infty\right) \quad ... (2)$$

$$(1)\cap(2) \Rightarrow x\in\left(\sqrt{\frac{2}{3}},\infty\right) \quad .... (3)$$

Case II : For $x < 0$ ... (4) $f'(2x^2 - 1) < f'(1 - x^2)$
$\Rightarrow \quad 2x^2 - 1 < 1 - x^2$

$$\Rightarrow \quad x\in\left(-\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}}\right) \quad ... (5)$$

$$(4)\cap(5) \Rightarrow x\in\left(-\sqrt{\frac{2}{3}},0\right) \quad ... (6)$$

$\therefore$ g is increasing in $x \in$ (3) $\cup$ (6)

$$\Rightarrow \quad x\in\left(-\sqrt{\frac{2}{3}},0\right)\cup\left(\sqrt{\frac{2}{3}},\infty\right)$$

9. (b) : $A = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$, $B = \{y_1, y_2, y_3\}$
$f : A \to B$ is onto, then there exist $f(x) = y_2$
Exactly 3 elements of x is associated with y_2. This can be done in ${}^7C_3$ ways.
Remaining four elements of A is associated with 2 elements of B.
$\therefore$ No. of ways $= 2^4 - {}^2C_1(2-1)^4 = 14$
$\therefore$ Total no. of onto functions $= {}^7C_3 \times 14$

10. (a) : $\int_0^3 \left(\sqrt{(x+3)+2\sqrt{3}\sqrt{x}} + \sqrt{(x+3)-2\sqrt{3}\sqrt{x}}\right)dx$

$\int_0^3 ((\sqrt{x}+\sqrt{3})+(\sqrt{3}-\sqrt{x}))dx = \int_0^3 2\sqrt{3}\,dx = 6\sqrt{3}$

11. (c) : $\frac{1}{x}+\frac{1}{y}=\frac{1}{2007} \Rightarrow (x+y)2007 = xy$

$\Rightarrow$ $xy - 2007x - 2007y = 0$
$\Rightarrow$ $(x-2007)(y-2007) = 2007^2 = 3^4 \times 223^2$
The number of pairs is equal to the number of divisors of 2007^2 that is $(4+1)\times(2+1) = 15$.
Since $x < y$, so required number of pairs $= 7$

12. (a) : Let α be a root of $f(x) = 0$, so we have $f(\alpha) = 0$ and thus $f(f(\alpha)) = 0$,
$\Rightarrow$ $f(0) = 0 \Rightarrow \mu = 0$
Then we have, $f(x) = x(x+\lambda)$ and thus $\alpha = 0, -\lambda$.
$f(f(x)) = x(x+\lambda)(x^2+\lambda x+\lambda)$
We want λ such that $x^2+\lambda x+\lambda$ has real roots.
$\Rightarrow$ $0 \le \lambda \le 4$
We can easily find that $0 \le \lambda < 4$.

13. (c) : Taking $\log_b$ on both sides, we get
$(\log_a b)^x \log_b a = (\log_b a)^x$
$\therefore$ $(\log_a b)^x = (\log_b a)^{x-1}$
$\therefore$ $1 - x = x \Rightarrow x = \frac{1}{2}$

14. (c) : The given function has local maximum at $x = -1$, minimum at $x = 0$ and $f(0) = 0$, $f(-1) = 1$,
$f(-3) = 2 - \sqrt[3]{3}$, $f(2) = 2^{2/3} = \sqrt[3]{4}$
$\therefore$ Range of $f(x) = [0, \sqrt[3]{4}]$

15. (b) : Let the points of intersection be $\left(t_1, \frac{1}{t_1}\right)$ and $\left(t_2, \frac{1}{t_2}\right)$.
Given $m = -\frac{1}{t_1 t_2}$ or $t_1 t_2 = -\frac{1}{m}$
Also by section formula, $t_2 + 2t_1 = 3x$
$t_1 + 2t_2 = -\frac{3y}{m}$
Solving for t_1, t_2 and eliminating them gives $2m^2x^2 + 5mxy + 2y^2 = m$, which is always a hyperbola as $\frac{25m^2}{4} - 4m^2 = \frac{9m^2}{4} > 0, \forall m \ne 0$

16. (a) : $\int \frac{2-3\sin^2 x}{\sqrt{\cos x}}dx = \int \frac{2\cos^2 x - \sin^2 x}{\sqrt{\cos x}}dx$
$= 2\int (\cos x)\sqrt{\cos x}\,dx - \int \frac{\sin^2 x}{\sqrt{\cos x}}dx$
$= 2\sin x\sqrt{\cos x} + c$
$\Rightarrow f(x) = \sin x, g(x) = \cos x$

17. (b) : a, b are factors of the form
$2^{a_1}5^{b_1}11^{c_1}$, $2^{a_2}5^{b_2}11^{c_2}$
where $a_1, b_1, c_1, a_2, b_2, c_2$ are non negative integers.
Since L.C.M of a, b is $2^3 5^7 11^{13}$, $\max\{a_1, a_2\} = 3$, $\max\{b_1, b_2\} = 7$ and $\max\{c_1, c_2\} = 13$.
Hence (a_1, a_2) can be (0, 3), (1, 3), (2, 3), (3, 3), (3, 2), (3, 1), (3, 0) (one of the number is 3 and other number can be any where from 0 to 3) giving us 7 choices. Similarly (b_1, b_2) has 15 choices and (c_1, c_2) has 27 choices.
Hence, total number of choices $= 7 \times 15 \times 27$
$= 2835$

18. (b) : Required value is
$\frac{{}^{10}C_1}{1} - \frac{{}^{10}C_2}{2} + \frac{{}^{10}C_3}{3} - \ldots - \frac{{}^{10}C_{10}}{10}$
consider
$(1-x)^{10} = {}^{10}C_0 - {}^{10}C_1 x + {}^{10}C_2 x^2 - \ldots + {}^{10}C_{10}x^{10}$
$\Rightarrow \frac{(1-x)^{10}-1}{x} = -[{}^{10}C_1 - {}^{10}C_2 x + \ldots - {}^{10}C_{10}x^9]$
$\Rightarrow \int_0^1 \frac{1-(1-x)^{10}}{x}dx = \int_0^1 [{}^{10}C_1 - {}^{10}C_2 x + \ldots - {}^{10}C_{10}x^9]dx$
$= {}^{10}C_1 - \frac{{}^{10}C_2}{2} + \frac{{}^{10}C_3}{3} - \ldots - \frac{{}^{10}C_{10}}{10}$
To find L.H.S
consider
$I_n = \int_0^1 \frac{1-(1-x)^n}{x}dx \Rightarrow I_{n+1} - I_n = \int_0^1 (1-x)^n dx = \frac{1}{n+1}$

$\therefore \quad I_{n+1} = \frac{1}{n+1} + I_n$

$\therefore \quad I_{10} = \int_0^1 \frac{1-(1-x)^{10}}{x} dx = \frac{1}{10} + I_9 = \frac{1}{10} + \frac{1}{9} + I_8 \approx \ldots$

$= \frac{1}{10} + \frac{1}{9} + \frac{1}{8} + \ldots + 1$

19. (c) : Number of ways of choosing a and b s.t $a \le b = 10 + 9 + 8 + \ldots + 1 = 55$

No. of ways of choosing a and b s.t a divides b

$= 10 + 5 + 3 + 2 + 2 + 5 = 27$

$\therefore$ Required probability $= \frac{27}{55}$

20. (d) : Area $= 2\int_{-2}^{0} y\, dx = 2\int_{-2}^{0} x^2\sqrt{x+2}\, dx$

$= 4\int_0^{\sqrt{2}} (z^2-2)^2 z^2 dz \quad (\text{where } \sqrt{x+2} = z)$

$= 4\left[\frac{z^7}{7} - \frac{4z^5}{5} + \frac{4z^3}{3}\right]_0^{\sqrt{2}} = \frac{256\sqrt{2}}{105}$

21. (b) : $\frac{6^k}{(3^k - 2^k)(3^{k+1} - 2^{k+1})}$

$= \frac{3^k 2^k}{(3^k-2^k)(3^{k+1}-2^{k+1})} = \frac{3^k}{3^k-2^k} - \frac{3^{k+1}}{3^{k+1}-2^{k+1}}$

$\Rightarrow \quad \sum_{k=1}^{\infty}\left(\frac{3^k}{3^k-2^k} - \frac{3^{k+1}}{3^{k+1}-2^{k+1}}\right) = \frac{3}{3-2} - \lim_{n\to\infty}\frac{3^n}{3^n-2^n}$

$= 3 - 1 = 2$

22. (c) : Coefficient of x^6 is

$\frac{1}{\lfloor 5}\frac{1}{\lfloor 1} + \frac{1}{\lfloor 4}\frac{1}{\lfloor 2} + \frac{1}{\lfloor 3}\frac{1}{\lfloor 3} + \frac{1}{\lfloor 2}\frac{1}{\lfloor 4} + \frac{1}{\lfloor 1}\frac{1}{\lfloor 5}$

$= \frac{1}{\lfloor 6}[{}^6C_1 + {}^6C_2 + {}^6C_3 + {}^6C_4 + {}^6C_5]$

$= \frac{1}{\lfloor 6}(2^6 - 2) = \frac{31}{360}$

23. (b) : $I = 4\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx = 4\int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x} dx$

$\therefore \quad 2I = 4\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx = 8\pi\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x} dx = 2\pi^2$

$\therefore \quad I = \pi^2$

24. (b) : a, b, c are in A.P. $\Rightarrow \quad a + c = 2b$ ... (1)

b, c, d are in G.P. $\Rightarrow \quad c^2 = bd$ ... (2)

c, d, e are in H.P. $\Rightarrow \quad \frac{2ce}{c+e} = d$ ... (3)

(1) $\times$ (3) $\Rightarrow \quad \frac{(a+c)ce}{c+e} = bd = c^2$

$\therefore \quad (a + c)e = c(c + e)$

$\Rightarrow \quad ae = c^2 \quad \Rightarrow \quad a, c, e$ are in G.P.

25. (c) : $a_1 = 1 \quad \Rightarrow \quad a_2 = 1 - \frac{1}{2}$

$\Rightarrow a_3 = 1 - \frac{1}{2} + \frac{1}{3}$ .....

$\lim_{n\to\infty}(a_n) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots = \ln 2$

26. (a) : $f'(x) = \frac{1}{2+x^4}$

By L.M.V.T. $f'(c) = \frac{f(2) - f(1)}{2-1}$ for some $c \in (1, 2)$

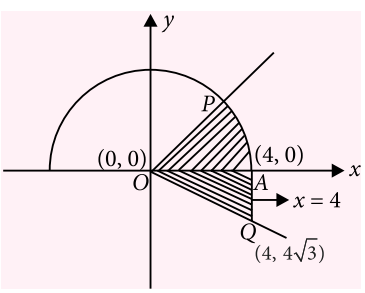
$\Rightarrow \quad f(2) = \frac{1}{2+c^4}$ as $f(1) = 0$

Since $1 < c < 2 \Rightarrow 3 < 2 + c^4 < 18 \Rightarrow f(2) < 1/3$

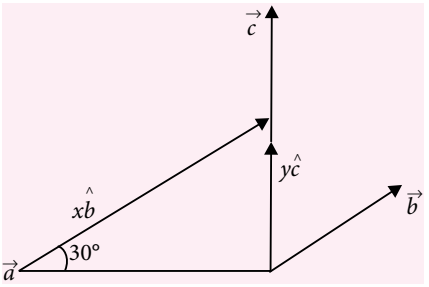
27. (b) : Required area is the area of shaded region $(APOQ)$

= area of ΔOAQ + area of sector (OAP)

$= \frac{1}{2}\times 4 \times 4\sqrt{3} + \frac{\pi(4\times 4)}{6} = \left(\frac{8\pi}{3} + 8\sqrt{3}\right)$



28. (c) : $\vec{a} = y\hat{c} - x\hat{b}$



$$\frac{x}{|\vec{a}|} = \sec 30° \Rightarrow x = \frac{2}{\sqrt{3}}|\vec{a}|$$

$$\frac{y}{|\vec{a}|} = \tan 30° \Rightarrow y = \frac{1}{\sqrt{3}}|\vec{a}|$$

$$\Rightarrow \vec{a} = \frac{1}{\sqrt{3}}\vec{c} - \frac{2}{\sqrt{3}}\vec{b}$$

29. (c) : $(2n + 1)^2 - (2m + 1)^2$
$= 4(m + n + 1)(n - m)$ = multiple of 8

30. (c) : P is centroid of ΔABC
$\therefore$ Area of $\Delta ABC = 6 \times 5 = 30$ sq.units

31. (c, d) : Let the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

For non-existence of $\perp^r$ tangents $b^2 > a^2 \Rightarrow e > \sqrt{2}$

32. (a, b) : $(\sin\alpha + \sin\beta)^2 = 1 \Rightarrow \sin\alpha + \sin\beta = -1$

$$\Rightarrow \sin\alpha = \frac{-1}{2} \text{ and } \sin\beta = \frac{-1}{2} \Rightarrow \alpha = \beta = -30°$$

33. (a, b) : When $x < 1$, $f(1) = \frac{h(1)+7}{3+1}$

$$\Rightarrow 7 = \frac{h(1)+7}{4} \Rightarrow h(1) = 21$$

When $x > 1$, $f(1) = \lim_{p\to\infty} \frac{\left[g(x) + \frac{h(x)}{x^p} + \frac{7}{x^p}\right]}{\left[7 + \frac{3x+1}{x^P}\right]}$

$$\Rightarrow 7 = \frac{g(1)}{7} \Rightarrow g(1) = 49$$

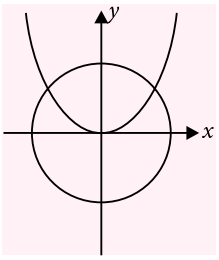
$\therefore$ $g(1) - h(1) = 28 \Rightarrow g(1) + h(1) = 70$

34. (a, b, c, d) : $a^2 = b^2(1 - e^2) \Rightarrow (\sec^2\alpha)e^2 = 1$

$\Rightarrow$ $e = \cos\alpha$ Also, $l = \frac{2a^2}{b}$

35. (a, b) : $x^2 + y^2 \le 25$ and

$9y \ge 4x^2 \Rightarrow x^2 \le \frac{9}{4}y$



Substituting $x^2 = \frac{9y}{4}$ in

$x^2 + y^2 = 25$, we get

$$\frac{9y}{4} + y^2 - 25 = 0$$

$\Rightarrow$ $4y^2 + 9y - 100 = 0$
$\Rightarrow$ $(y - 4)(4y + 25) = 0 \Rightarrow y = 4$
Domain and range of $R \cap R'$ are $[-3, 3]$ and $[0, 4]$

36. (a, b, c) : $f(1 + h) = 2 = f(1 - h)$
$f(3^+) = f(3^-) = 0 \Rightarrow$ Continuous at $x = 1, 3$
$f'(1^+) = f'(1^-) = -1$
Not differentiable at $x = 3$

37. (c) : $\lim_{x\to\infty} \frac{\tan^{-1}\left(\frac{x+\lambda}{x+\mu}\right) - \frac{\pi}{4}}{\frac{1}{x}} = 1$

Apply L' hospital rule and simplifying, we get

$$\lim_{x\to\infty} \frac{(\lambda - \mu)x^2}{2x^2 + 2x(\lambda + \mu) + (\mu^2 + \lambda^2)} = 1$$

$$\Rightarrow \frac{\lambda - \mu}{2} = 1 \Rightarrow \lambda - \mu = 2 \therefore (\lambda, \mu) \text{ can be } (5, 3)$$

38. (a, b, c) : If the sides are a,a,b then the triangle forms only when $2a > b$. So for any $a \in N$, b can change from 1 to $2a - 1$, where $a \le 1004$

$\Rightarrow$ Number of triangles
$= 1 + 3 + 5 \ldots\ldots + (2(1004) - 1) = (1004)^2$
And if $1005 \le a \le 2008$, b can take any value from 1 to 2008
But a has 1004 possibilities.
Hence, no. of triangles $= 1004 \times 2008 = 2(1004)^2$

$\therefore$ Total no. of isosceles triangles $= 3(1004)^2$

39. (a, b, c, d) : $\tan\frac{B}{2} = \frac{2}{3}, \tan\frac{C}{2} = \frac{1}{2}$

$$\Rightarrow \tan\frac{A}{2} = \frac{4}{7}$$

$$\tan\frac{B}{2}.\tan\frac{C}{2} = \frac{s-a}{s} \Rightarrow 2s = 3a = 3 \times 14 = 42$$

$\therefore$ Perimeter = 42
$\therefore$ $\Delta = r \cdot s = 84$ cm^2

$\because$ $\tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}$ all are less than 1.

$\therefore$ All angles are acute.

40. (a, b) : Let r be common ratio of G.P. we have,

$$z = \frac{\frac{a}{b}+i}{\frac{c}{b}-i} = \frac{\frac{1}{r}+i}{r-i} = \frac{i}{r} \quad \text{or} \quad z = \frac{ib}{c} \text{ or } \frac{ia}{b}$$

41. (b) : $(f'(x))^2 - k^2(f(x))^2 \le 0$

$\Rightarrow$ $(f'(x) - kf(x))(f'(x) + kf(x)) \le 0$

$\Rightarrow$ $(f(x)e^{-kx})(f(x)e^{kx}) \le 0$

$\Rightarrow$ Exactly one of the functions $g_1(x) = f(x)e^{-kx}$ or $g_2(x) = f(x)e^{kx}$ is non decreasing.

But $f(0) = 0$ $\Rightarrow$ both function g_1 and g_2 have a value zero at $x = 0$

42. (b, c, d) : Solving, we get

$x = 2^{1/2}$, $y = 3^{1/3}$, $z = 5^{1/5}$

$3^{1/3} > 5^{1/3}$ as $3^5 > 5^3$

Also, $2^{1/2} < 3^{1/3}$ as $2^3 < 3^2$ and $2^{1/2} > 5^{1/5}$ as $2^5 > 5^2$

$\Rightarrow$ $y > x > z$

Hence (b), (c), & (d) are NOT true.

43. (a, c, d) : $T_r = \frac{r(\sqrt{r+1}) - (r+1)\sqrt{r}}{r^2(r+1) - (r+1)^2 r}$

$$= \frac{r\sqrt{r+1} - (r+1)\sqrt{r}}{-r^2 - r} = \frac{(r+1)\sqrt{r}}{r(r+1)} - \frac{r\sqrt{r+1}}{r(r+1)}$$

$$= \frac{1}{\sqrt{r}} - \frac{1}{\sqrt{r+1}}$$

$$\Rightarrow \sum_{r=1}^{99} T_r = \frac{1}{1} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \ldots\ldots - \frac{1}{\sqrt{100}} = 1 - \frac{1}{\sqrt{100}} = \frac{9}{10}$$

Hence (a), (c) and (d) are correct.

44. (a, d) : $S_1 : 1$, $S_2 : 1 + \frac{1}{2}$, $S_3 : 1 + \frac{1}{2} + \frac{1}{3}$

$$S_{n-1} : 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{n-1}$$

Adding

$$= (n-1) + \frac{(n-2)}{2} + \frac{(n-3)}{3} + \ldots\ldots \left(\frac{n-(n-1)}{(n-1)}\right)$$

$$= n\left[1 + \frac{1}{2} + \frac{1}{3} + \ldots. + \frac{1}{n-1}\right] - [1+1+1+\ldots\ldots+1]$$

$= nS_{n-1} - (n-1) = nS_n - n$

45. (a, c) : $P(E \cap F) = P(E) \cdot P(F) = \frac{1}{6}$ ...(i)

$$P(E^c \cap F^c) = (1 - P(E))(1 - P(F)) = \frac{1}{3}$$

$\Rightarrow$ $P(E) + P(F) = \frac{5}{6}$ ... (ii)

As $(P(E) - P(F))(1 - P(F)) > 0$ $\Rightarrow$ $P(E) > P(F)$

Solving (i) and (ii), we get $\Rightarrow$ $P(E) = \frac{1}{2}$, $P(F) = \frac{1}{3}$

46. (c) : $\phi(1) = 1$, $\phi(p^n) = p^{n-1}(p-1)$

$\phi(mn) = \phi(m) \cdot \phi(n)$

$\phi(8n + 4) = \phi(4(2n + 1)) = \phi(4) \cdot \phi(2n + 1)$

$= \phi(2^2) \cdot \phi(2n + 1) = 2 \cdot \phi(2n + 1)$

47. (b) : $\phi(n)$ is odd. $\Rightarrow$ $\phi(p^n)$ is odd

$\Rightarrow$ $p^{n-1}(p-1)$ is odd.

$\because$ p is prime. The only value p can take is $p = 2$

$\therefore$ $\phi(2^n)$ is odd.

$\Rightarrow$ $2^{n-1}(2-1) = 2^{n-1}$ is odd.

$\Rightarrow$ $n - 1 = 0$ $\Rightarrow$ $n = 1$ $\therefore$ $\phi(1) = 1 = \phi(2)$

48. (b) : $\phi(7^n) = 2058$

$7^{n-1}(7-1) = 2058$

$7^{n-1} = 343$ or $n - 1 = 3$ $\therefore$ $n = 4$

(49-51) :

49. (d) 50. (b) 51. (c)

For the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$, $|A - \lambda I| = 0$

$\Rightarrow$ $\lambda^3 - 5\lambda^2 + \lambda + 3 = 0$ ...(1)

whose roots are $\lambda_1, \lambda_2, \lambda_3$

$\lambda_1 + \lambda_2 + \lambda_3 = 5$, $\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = 1$, $\lambda_1\lambda_2\lambda_3 = -3$

$\Rightarrow$ $Tr(A) = 5$, $\det(A) = -3$

Also, $AX = \lambda X$ $\Rightarrow$ $A^{-1}X = \frac{1}{\lambda}X$

$$\Rightarrow Tr(A^{-1}) = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = \frac{\sum \lambda_1\lambda_2}{\lambda_1\lambda_2\lambda_3} = -\frac{1}{3}$$

$$AX = \lambda X \Rightarrow A^3X = \lambda^3 X$$

$$\Rightarrow Tr(A^3) = \lambda_1^3 + \lambda_2^3 + \lambda_3^3$$

$= (\lambda_1 + \lambda_2 + \lambda_3)(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - \lambda_1\lambda_2 - \lambda_2\lambda_3 - \lambda_3\lambda_1) + 3\lambda_1\lambda_2\lambda_3$

$= 5(25 - 3(1)) - 9 = 101$

Solving (1) gives $\lambda_1 = 1,\ \lambda_2 = 2+\sqrt{7}, \lambda_3 = 2-\sqrt{7}$, which by theory yield non-trivial solutions. In particular for A^{-1}, the values of λ yield non trivial solutions are 1, $\frac{1}{2+\sqrt{7}}, \frac{1}{2-\sqrt{7}}$ *i.e.* 1, $\frac{2-\sqrt{7}}{-3}, \frac{2+\sqrt{7}}{-3}$

Hence (c) is false.

(52-54) :

52. (b) 53. (a) 54. (c)

Since the net movement must be two steps right (R) and two steps up (U) there must be atleast 4 steps to reach (2, 2) in 6 or fewer steps. (2, 2) can be reached in 4 steps if the sequence of steps is some permutations of R, R, U, U

$\therefore$ Probability of reaching (2, 2) in 4 steps

$$= \frac{\left(\frac{4!}{2!2!}\right)}{4^4} = \frac{6}{4^4} = \frac{3}{128}$$

A six step sequence moludes the steps R, R, U, U in same order as well as a pair of steps consisting of R, L or U, D in same order. R, R, U, U, R, L or R, R, U, U, U, D can be permuted in 2(60) ways of which 2 × 12 correspond to exactly 4 steps.

Hence probability for exactly six steps

$$= \frac{2(60-12)}{4^6} = \frac{6}{4^4}$$

Probability for six or fewer steps $= \frac{6}{4^4} + \frac{6}{4^4} = \frac{3}{64}$

55. (a) : In $g(m, n)$, putting $x = \frac{y}{1+y}$, we get $dx = \frac{1}{(1+y)^2}dy$ and we get

$$g(m, n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m-1}} \cdot \frac{1}{(1+y)^{n-1}} \cdot \frac{1}{(1+y)^2} dy$$

$$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

56. (c) : Putting $\log_e \frac{1}{x} = t \quad \Rightarrow \quad x = e^{-t}$

Now, $\int_0^1 x^m \left(\log_e \frac{1}{x}\right)^n dx$

$$= \int_\infty^0 e^{-mt} t^n (-e^{-t})\, dt = \int_0^\infty t^n e^{-(m+1)t} dt$$

$$= \frac{1}{(m+1)^{n+1}} \int_0^\infty t^n e^{-t} dt = \frac{f(n+1)}{(m+1)^{n+1}}$$

57. (a) : $I = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$$= I_1 + I_2$$

In I_2, put $x = 1/t \Rightarrow \quad I_2 = \int_\infty^1 \frac{\frac{1}{t^{n-1}}}{\left(1+\frac{1}{t}\right)^{m+n}} \left(\frac{-dt}{t^2}\right)$

$$= \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

$\therefore \quad I = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$

$$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = g(m, n)$$

58. (b) : $\sum_{r=0}^{200} a_r \left(\frac{1}{x}\right)^r = \left(1 + \frac{1}{x} + \frac{1}{x^2}\right)^{100}$

$$= \frac{1}{x^{200}} (x^2 + x + 1)^{100}$$

$$\Rightarrow \quad \sum_{r=0}^{200} a_r x^{200-r} = (x^2 + x + 1)^{100}$$

$$\Rightarrow \quad \sum_{r=0}^{200} a_r x^r = \sum_{r=0}^{200} a_{200-r} x^r$$

Equating the coefficients of x^{200-r}, we get $a_r = a_{200-r}$

59. (c) : Put $x = 1$

$\Rightarrow \quad a_0 + a_1 + a_2 + \ldots + a_{200} = 3^{100}$

But $a_r = a_{200-r}$

$\therefore \quad 2(a_0 + \ldots + a_{99}) + a_{100} = 3^{100}$

$$\Rightarrow \quad a_0 + a_1 + \ldots + a_{99} = \frac{1}{2}(3^{100} - a_{100})$$

60. (a) : Differentiating, we get

$$100(1+2x)(1+x+x^2)^{99} = \sum_{r=0}^{200} r a_r x^{r-1}$$

61. (A) → (p, q, r, t); (B) → (p, r, s); (C) → (q, t) ; (D) → (p, q, r, t)

(A) $\cos x + \sin^2 x$ has a local maximum at $x = \frac{\pi}{3}$ and $x = \frac{-\pi}{3}$ in the interval $(-\pi, \pi)$

(B) $\tan^{-1}(\sin x - \cos x)$ is strictly increasing in $\left(-\frac{\pi}{4}, \frac{3\pi}{4}\right)$

(C) $x^2 - 5x + 6 = \left(x - \frac{5}{2}\right)^2 - \frac{1}{4} \geq -\frac{1}{4}$ for all x

$\because\ 2 < x < 3 \Rightarrow \frac{-1}{4} \leq x^2 - 5x + 6 < 0$

$\Rightarrow\ [x^2 - 5x + 6] = -1\ \ \therefore\ \int_2^3 [x^2 - 5x + 6]\, dx = -1$

(D) $3I = \int_{-3}^{3} \frac{3}{3 + f(x)} = \int_{-3}^{3} \frac{3}{3 + f(-x)} dx$

$= \int_{-3}^{3} \frac{3f(x)}{3f(x) + 9} dx = \int_{-3}^{3} \frac{f(x)}{3 + f(x)} dx$

$\Rightarrow\ 3I = \int_{-3}^{3} \frac{(f(x) + 3) - 3}{f(x) + 3} dx = \int_{-3}^{3} 1\, dx - \int_{-3}^{3} \frac{3}{3 + f(x)} dx$

$\therefore\ \ I = 1$

62. (A) → (q); (B) → (q); (C) → (r); (D) → (t)

(A) $I = \int_0^{\pi} x(\sin^2(\sin x) + \cos^2(\cos x))dx$

$\Rightarrow\ I = \int_0^{\pi} (\pi - x)(\sin^2(\sin x) + \cos^2(\cos x))dx$

Adding, $2I = \pi \int_0^{\pi} (\sin^2(\sin x) + \cos^2(\cos x))dx$

$\Rightarrow\ \ 2I = 2\pi \int_0^{\pi/2} (\sin^2(\sin x) + \cos^2(\cos x))dx$

$\Rightarrow\ \ I = \pi \int_0^{\pi/2} (\sin^2(\sin x) + \cos^2(\cos x))dx$

Also, $I = \pi \int_0^{\pi/2} \left[\sin^2(\cos x) + \cos^2(\sin x)\right] dx$

Adding, $2I = \pi \int_0^{\pi/2} 2dx = 2\pi \cdot \pi/2 = \pi^2 \ \Rightarrow\ I = \pi^2/2$

(B) Let $f(x) = 2\sin\sqrt{x}$

Then, $x\, f'(x) = x.\frac{\cos\sqrt{x}}{\sqrt{x}} = \sqrt{x}\cos\sqrt{x}$

$I = \int_0^{\pi^2/4} (f(x) + x f'(x))\, dx = \left[x\ f(x)\right]_0^{\pi^2/4}$

$= 2 \times \frac{\pi^2}{4} \times \sin\frac{\pi}{2} = \frac{\pi^2}{2}$

(C) $\int_{-\pi/4}^{\pi/4} \ln\sqrt{1 + \sin 2x}\ dx = \int_{-\pi/4}^{\pi/4} \ln|\sin x + \cos x|\, dx$

$= \int_{-\pi/4}^{\pi/4} \ln|\sqrt{2}\ \sin(x + \pi/4)|\, dx$

$= \int_0^{\pi/2} \ln|\sqrt{2}\ \sin t|\, dt = \int_0^{\pi/2} \ln\sqrt{2}\ dt + \int_0^{\pi/2} \ln\sqrt{2}\ \sin t\ dt$

$= \ln\sqrt{2} \int_0^{\pi/2} dt + \int_0^{\pi/2} \ln\sin t\ dt$

$= \frac{\pi}{4}\ln 2 - \frac{\pi}{2}\ln 2 = -\frac{\pi}{4}\ln 2$

$\therefore\ \ \frac{\pi}{\ln 2} \int_{-\pi/4}^{\pi/4} \ln\sqrt{1 + \sin^2 x}\, dx = \left(\frac{\pi}{\ln 2}\right)\left(\frac{-\pi}{4}\ln 2\right) = \frac{-\pi^2}{4}$

(D) $I = \int_0^{\pi} \frac{x^3 \cos^4 x \sin^2 x}{\pi^2 - 3\pi x + 3x^2}\, dx$

$I = \int_0^{\pi} \frac{(\pi - x)^3 \cos^4 x \sin^2 x}{\pi^2 - 3\pi x + 3x^2}\, dx$

Adding $2I = \pi \int_0^{\pi} \cos^4 x \sin^2 x\ dx$

$\Rightarrow\ \ I = \frac{\pi}{2}.2\int_0^{\pi/2} \cos^4 x \sin^2 x\, dx = \pi \int_0^{\pi/2} \cos^4 x \sin^2 x\ dx$

$= \pi \cdot \frac{(3 \cdot 1)}{6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} = \frac{\pi^2}{32}$

63. (A) → (q); (B) → (r); (C) → (s); (D) → (p)

(A) Required number of selection

$= {}^{2n}C_0 + {}^{2n}C_1 + \ldots + {}^{2n}C_n = 2^{2n-1} + \frac{1}{2}\frac{(2n)!}{(n!)^2}$

(B) ${}^nC_4$ (each quadrilateral gives one point of intersection)

(C) $x_4 > x_3 > x_2 > x_1 > x_0$

${}^{10}C_5$ (5 distinct digits selection)

(D) Terms is involving 3^0, $3^{1/3}$, $3^{2/3} \rightarrow 3$

64. (A) → (r); (B) → (q); (C) → (s); (D) → (p)

(A) ABC is pedal triangle of acute angle $l_1l_2l_3$ and of obtuse angled triangle ll_1l_2, ll_2l_3 and ll_3l_1.

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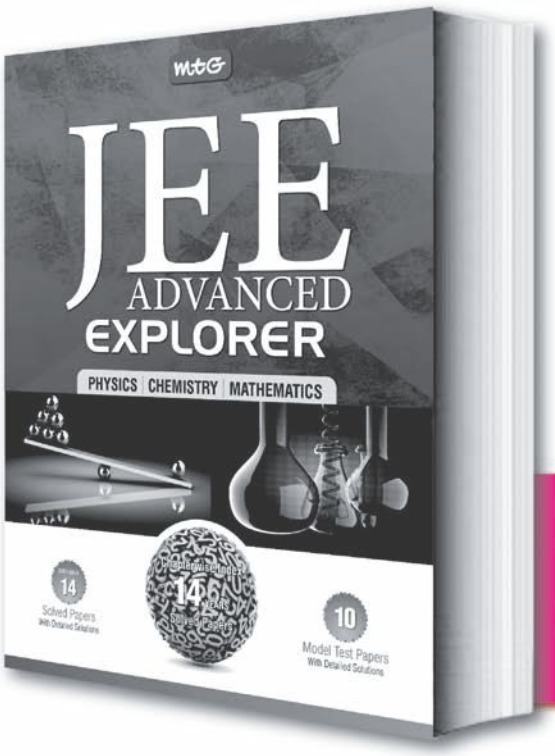

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(B) $\frac{R_2}{R_1}=2$

(C) In ΔABC, if AD, BE and CF are concurred and if $\frac{BD}{DC}=\frac{\alpha}{\beta}$ and $\frac{CE}{AE}=\frac{\beta}{\gamma}$

Then $\frac{AF}{FB}=\frac{\gamma}{\alpha}$

$\Rightarrow$ $BD \cdot CE \cdot AF = DC \cdot AE \cdot FB$

(D) $\frac{3}{2}\left(1-\frac{s-b}{s-a}\right)\left(1-\frac{s-c}{s-a}\right)$

65. (A) → (p); (B) → (r); (C) → (r); (D) → (s)

(A) Two tangent can be drawn because curve $y = ||1 - e^x| - 2|$ intersect the line $y = 1$ at two points

(B) $(x - 1)^2 + (y - 2)^2 = 3^2 \Rightarrow a = 3$

(C) $PQ = a + x = 2 + 1 = 3$

(D) Locus of focus of parabola is $2x^2 - 2x + 2y^2 - 2y + 1 = 0$

$\therefore$ radius is zero

66. (8) : $f'(x)=\lim\limits_{h\to 0}\frac{f(x+h)-f(x+0)}{h}$

$\Rightarrow \lim\limits_{h\to 0}\frac{f(x)+f(h)+h\sqrt{f(x)}-f(x)-f(0)-0\sqrt{f(x)}}{h}$

$\Rightarrow \lim\limits_{h\to 0}\left(\frac{f(h)-f(0)}{h-0}\right)+\sqrt{f(x)} \Rightarrow f'(x)=\sqrt{f(x)}$

$\int\frac{f'(x)}{\sqrt{f(x)}}dx=\int dx \Rightarrow 2\sqrt{f(x)}=x+c$

$f(x)=\frac{x^2}{4}$ $\therefore$ When $\alpha = 0$ area is minimum

Required minimum area $= 2\int\limits_0^9 2\sqrt{y}\,dy$

$\Rightarrow 4\left.\frac{y^{3/2}}{3/2}\right|_0^9 = 72$ sq. units

67. (5) : Let Y_i be the subset of X such that $Y_i = 7m + i$, $m \in I$

$Y_0 = \{7, 14, ..., 98\}$, $n(Y_0) = 14$

$Y_1 = \{1, 8, 15, ..., 99\}$, $n(Y_1) = 15$

$Y_2 = \{2, 9, 16, ..., 100\}$, $n(Y_2) = 15$

$Y_3 = \{3, 10, 17, ..., 94\}$, $n(Y_3) = 14$

$Y_4 = \{4, 11, 18, ..., 95\}$, $n(Y_4) = 14$

$Y_5 = \{5, 12, ..., 96\}$, $n(Y_5) = 14$

$Y_6 = \{6, 13, ..., 97\}$, $n(Y_6) = 14$

The largest Y will consist of (i) an element of Y_0 (ii) Y_1 (iii) Y_2 (iv) Y_3 or Y_4

$\Rightarrow$ The maximum possible number of elements in $Y = 1 + 15 + 15 + 14 = 45$.

68. (2) : $\sin(2\sin x)=\sin\left(\frac{\pi}{2}-2\cos x\right)$

$\sin x+\cos x=\frac{\pi}{4} \Rightarrow 1+\sin 2x=\frac{\pi^2}{16}$

$\sin 2x=\frac{\pi^2-16}{16}$

$\therefore \tan x+\cot x=\frac{2}{\sin 2x}=\frac{2\times 16}{\pi^2-16}=\frac{32}{\pi^2-16}$

$\therefore$ $a = 32$, $b = 16$, $c = 2$ $\Rightarrow \frac{a+b+c}{25}=2$

69. (1) : Solving the equation of the lines we get $z=-\bar{z}\Rightarrow z=i$

$|\alpha-i|=2$; $\alpha=2e^{i\theta}+i$, put it in the equation of the second line, we get

$\cos\theta - \sin\theta = 0 \Rightarrow \alpha=i\pm 2e^{\frac{i\pi}{4}}$

$\therefore$ $x=\pm\sqrt{2} \Rightarrow [|x|]=1$

70. (2) : Let P be $(5\cos\theta, 4\sin\theta)$ and Q be $(-5\sin\theta, 4\cos\theta)$

Equation of tangent at P is

$$\frac{x}{5}\cos\theta+\frac{y}{4}\sin\theta=1 \qquad \text{.... (i)}$$

Equation of tangent at Q is

$$-\frac{x}{5}\sin\theta+\frac{y}{4}\cos\theta=1 \qquad \text{....(ii)}$$

Solving (i) and (ii), we get

$\Rightarrow R=(5(\cos\theta-\sin\theta), 4(\sin\theta+\cos\theta))$

$\therefore$ $m : n$ is $1 : 1$ $\Rightarrow m + n = 2$

71. (7) : Let $I=\int\limits_0^1(1-x^4)^7\cdot 1\,dx$

$=[x(1-x^4)^7]_0^1+7\times 4\int\limits_0^1 x(1-x^4)^6x^3dx$

$=-28\int\limits_0^1(1-x^4)^7dx+28\int\limits_0^1(1-x^4)^6dx$

$$\Rightarrow \quad I = -28I + 28\int_0^1 (1-x^4)^6\,dx$$

$$\Rightarrow \quad 29I - 28\int_0^1 (1-x^4)^6\,dx = 0$$

$$\Rightarrow \quad \frac{29\int_0^1 (1-x^4)^7\,dx}{4\int_0^1 (1-x^4)^6\,dx} = 7$$

72. (6) : $f'(x) = -a\pi\sin(\pi x)$

$$\Rightarrow f'\left(\frac{1}{2}\right) = -a\pi\sin\frac{\pi}{2} = -a\pi = \pi \Rightarrow a = -1$$

$$\int_{1/2}^{3/2} (a\cos\pi x + b)dx = \left[\frac{a\sin\pi x}{\pi} + bx\right]_{1/2}^{3/2}$$

$$= \left(\frac{-a}{\pi} + \frac{3b}{2}\right) - \left(\frac{a}{\pi} + \frac{b}{2}\right)$$

$$\Rightarrow \quad \frac{-2a}{\pi} + b = \frac{2}{\pi} + 1 \Rightarrow b = 1$$

So, $\frac{-12}{\pi}(\sin^{-1}(-1) + \cos^{-1}1) = \frac{-12}{\pi}\left(-\frac{\pi}{2} + 0\right) = 6$

73. (0) : Since $[\sqrt{2046}] = [\sqrt{2047}] = [\sqrt{2048}]$

$= [\sqrt{2049}] = 45$

$\therefore$ 2003^{rd} term is 2003 + 45 = 2048

Hence remainder is 0.

74. (7) : Let us form that equation having roots $y = g(x)$ *i.e.* $y = x^2 - 2$

$$x = \sqrt{y+2} \quad \Rightarrow \quad (\sqrt{y+2})^5 + (\sqrt{y+2})^2 + 1 = 0$$

$$\Rightarrow \quad y^5 + 10y^4 + 40y^3 + 79y^2 + 74y + 23 = 0$$

$\therefore$ $g(x_1)$ .... $g(x_2)$ = Product of roots = -23

$$x_1x_2x_3x_4x_5 = -1 \Rightarrow g(x_1x_2.....x_5) = -1$$

$$\Rightarrow g(x_1)g(x_2).....g(x_5) - 30g(x_1x_2.....x_5) = 7$$

75. (7) : $f(x) + f(\alpha x) + f(\alpha^2 x) + ... + f(\alpha^6 x)$

$$= 7A_0 + \sum_{k=1}^{20} A_k x^k (1 + \alpha^k + ... + \alpha^{6k})$$

but when $k \neq 7$ and $k \neq 14$, then

$$1 + \alpha^k + \alpha^{2k} + ... + \alpha^{6k} = 0$$

Hence, $f(x) + f(\alpha x) + ... + f(\alpha^6 x)$

$= 7A_0 + 7A_7x^7 + 7A_{14}x^{14} = 7(A_0 + A_7x^7 + A_{14}x^{14})$

$\therefore$ $k = 7$

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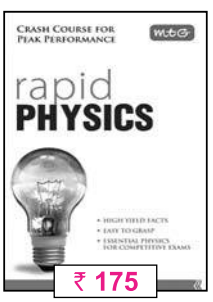

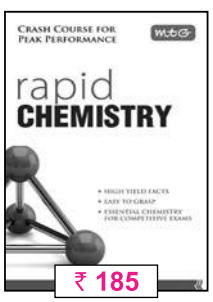

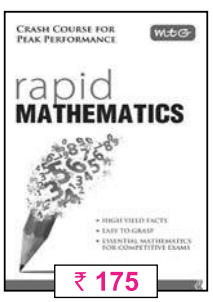

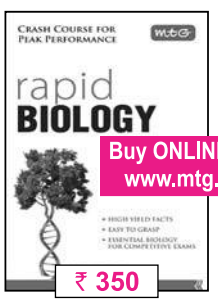

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Math Archives

10 Best Problems

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of JEE (Main & Advanced) Syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for JEE (Main & Advanced). In every issue of MT, challenging problems are offered with detailed solution. The readers' comments and suggestions regarding the problems and solutions offered are always welcome.

1. $\tan^2\frac{\pi}{16}+\tan^2\frac{2\pi}{16}+\tan^2\frac{3\pi}{16}+\tan^2\frac{5\pi}{16}+\tan^2\frac{6\pi}{16}+\tan^2\frac{7\pi}{16}$ is equal to

(a) 24 (b) 32
(c) 44 (d) none of these

2. If $f(x)=e^{\sin(x-[x])\cos\pi x}$, then $f(x)$ is ($[x]$ denotes the greatest integer function)

(a) non-periodic
(b) periodic with no fundamental period
(c) periodic with period 2
(d) periodic with period π

3. Which of the following homogeneous functions are of degree zero?

(a) $\frac{x}{y}\ln\frac{y}{x}+\frac{y}{x}\ln\frac{x}{y}$ (b) $\frac{x(x-y)}{y(x+y)}$
(c) $\frac{xy}{x^2+y^2}$ (d) all of the above

4. If θ is small and positive number, then which of the following is/are correct?

(a) $\frac{\sin\theta}{\theta}=1$ (b) $\theta<\sin\theta<\tan\theta$
(c) $\frac{\tan\theta}{\theta}>\frac{\sin\theta}{\theta}$ (d) none of these

5. Match the column. (Each entry of column I matches with exactly one entry of column II).

Column I	Column II
(A) $\lim_{x\to0}\log_{\sin x}\sin 2x$	(P) -1
(B) $\lim_{x\to0}\frac{\ln(\cos x)}{x}$	(Q) $-1/2$
(C) $\lim_{x\to1}\left(\frac{1}{\ln x}-\frac{x}{\ln x}\right)$	(R) 1
(D) $\lim_{x\to0}\frac{x-\sin x}{x-\tan x}$	(S) 0

(a) A → S, B → P, C → Q, D → R
(b) A → P, B → Q, C → R, D → S
(c) A → R, B → S, C → P, D → Q
(d) A → Q, B → R, C → S, D → P

6. If $y=\cos^{-1}\sqrt{\frac{\cos 3x}{\cos^3 x}}$, then prove that $\frac{dy}{dx}=\sqrt{\frac{3}{\cos x\cdot\cos 3x}}$.

7. Let $a\in R$, then prove that a function $f: R\to R$ is differentiable at a if a function $\phi: R\to R$ satisfies $f(x)-f(a)=\phi(x)(x-a)\ \forall\ x\in R$ and ϕ is continuous at a.

8. If $\beta,\gamma\in(0,\pi)$ such that $\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+\beta+\gamma)=0$ and $\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+\beta+\gamma)=0$. Then, evaluate $f'(\beta)$ and $\lim_{x\to\gamma}g(x)$, where $f(x)=\sin 2x(1+\cos 2x)^{-1}$ and $g(x)=\frac{1+\sin x-\cos x}{1+\sin x+\cos x}$

(Here $f'(x)$ denotes derivative of f with respect to x).

9. Find the area of the triangle formed with vertices $(0,0)$, $\left(\lim_{x\to\frac{\pi}{2}}\left[\frac{x-\frac{\pi}{2}}{\cos x}\right],0\right)$ and $\left(0,\lim_{x\to0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x}}\right)$ where $[\cdot]$ denotes the greatest integer function.

By : **Prof. Shyam Bhushan,** Director, Narayana IIT Academy, Jamshedpur. Mob. : 09334870021

10. Prove that the straight lines whose direction cosines are given by the relations $al + bm + cn = 0$ and $fmn + gnl + hlm = 0$ are perpendicular, if $\frac{f}{a}+\frac{g}{b}+\frac{h}{c}=0$.

SOLUTIONS

1. **(d)**

2. **(c) :** $f(x) = e^{\sin(x - [x])\cos\pi x}$

$\sin(x - [x]) = \sin\{x\}$, period is 1

$\cos\pi x$; Period is 2

Hence period of $f(x)$ is 2

3. **(d) :** $f(x, y)$ is homogeneous function of degree $n \in R$ in x, y if $f(kx, ky) = k^n f(x, y)$; where $k > 0$.

4. **(a)**

5. **(c)**

6. $\cos y = \sqrt{\frac{\cos 3x}{\cos^3 x}} = \sqrt{\frac{4\cos^3 x - 3\cos x}{\cos^3 x}}$ ... (i)

$= \sqrt{4 - 3\sec^2 x}$

$\Rightarrow \cos^2 y = 4 - 3\sec^2 x = 4 - 3(1 + \tan^2 x)$

$= 1 - 3\tan^2 x$

$\Rightarrow \sin^2 y = 3\tan^2 x \Rightarrow \sin y = \sqrt{3}\tan x$

$\Rightarrow \cos y \frac{dy}{dx} = \sqrt{3}\sec^2 x$

$\therefore \frac{dy}{dx} = \sqrt{\frac{3}{\cos x \cdot \cos 3x}}$ [$\because$ from (i)]

7. $\because f : R \to R$ is continuous at $x = a$ and satisfies $f(x) - f(a) = \phi(x)(x - a) \ \forall \ x \in R$

$\Rightarrow \frac{f(x) - f(a)}{x - a} = \phi(x)$

$\Rightarrow \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \phi(x)$

$\Rightarrow f'(a) = \phi(a)$ $\left[\because \lim_{x \to a} \phi(x) = \phi(a)\right]$

$\Rightarrow f$ is differentiable at $x = a$.

8. Given

$\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + \beta + \gamma) = 0$

$\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + \beta + \gamma) = 0$

where $\beta, \gamma \in (0, \pi)$

$\Rightarrow [\cos\alpha + \cos(\alpha + \beta)]^2 + [\sin\alpha + \sin(\alpha + \beta)]^2 = 1$

$\Rightarrow 2 + 2[\cos(\beta)] = 1$

$\therefore \cos\beta = -1/2$

Similarly, $\cos\gamma = -1/2$

$\therefore \beta = \gamma = \frac{2\pi}{3}$

But $f(x) = \frac{\sin 2x}{1 + \cos 2x} = \tan x$ and $g(x) = \tan\frac{x}{2}$

$\therefore f'\left(\frac{2\pi}{3}\right) = \sec^2\frac{2\pi}{3} = 4$

and $\lim_{x \to \frac{2\pi}{3}} g(x) = \tan\frac{\pi}{3} = \sqrt{3}$

9. Let $O = (0, 0)$

$A = \left(\lim_{x \to \frac{\pi}{2}} \left[\frac{x - \frac{\pi}{2}}{\cos x}\right], 0\right) = (-2, 0)$

$B = \left(0, \lim_{x \to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x}}\right) = (0, 1)$

$\therefore$ Area of $\Delta OAB = \frac{1}{2}|0 - 2 - 0| = 1$ square unit

10. $al + bm + cn = 0$ ...(1)

$fmn + gnl + hlm = 0$ ...(2)

Eliminating n, we get

$\Rightarrow ag\left(\frac{l}{m}\right)^2 + (af + bg - ch)\left(\frac{l}{m}\right) + bf = 0$ ...(3)

Now, if l_1, m_1, n_1 and l_2, m_2, n_2 are d.c's of two lines then roots of (3) are

$\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$

$\therefore$ Product of the roots $= \frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \frac{bf}{ag}$

$\therefore \frac{l_1 l_2}{f/a} = \frac{m_1 m_2}{g/b}$

$\therefore \frac{l_1 l_2}{f/a} = \frac{m_1 m_2}{g/b} = \frac{n_1 n_2}{h/c}$

$\because$ lines are perpendicular

$\therefore l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

$\Rightarrow \frac{f}{a}+\frac{g}{b}+\frac{h}{c}=0$

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JEE Main

Exam on 4th April

* ALOK KUMAR, B.Tech, IIT Kanpur

1. The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the axes at A, B, C. The orthocentre of the triangle must be at, where $k^4 = a^2b^2 + b^2c^2 + c^2a^2$

(a) $\left(\frac{ab^2c^2}{k^4}, \frac{ba^2c^2}{k^4}, \frac{ca^2b^2}{k^4}\right)$

(b) $\left(\frac{a^3}{k^2}, \frac{b^3}{k^2}, \frac{c^3}{k^2}\right)$

(c) $\left(\frac{a^2}{k}, \frac{b^2}{k}, \frac{c^2}{k}\right)$ (d) none of these

2. P is a point (a, b, c). Let A, B, C be images of P in yz, zx and xy planes respectively, then equation of the plane ABC must be

(a) $bcx + acy + abz = 3abc$

(b) $bcx + acy + abz = 2abc$

(c) $bcx + acy + abz = abc$

(d) $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$

3. The global maximum value of $f(x) = \log_{10}(4x^3 - 12x^2 + 11x - 3)$, $x \in [2, 3]$ is

(a) $-\frac{3}{2}\log_{10} 3$ (b) $1 + \log_{10} 3$

(c) $\log_{10} 3$ (d) $\frac{3}{2}\log_{10} 3$

4. Let $f(x)$ be a twice differentiable function for all real value of x and satisfies $f(1) = 1, f(2) = 4, f(3) = 9$, then which of the following is definitely true?

(a) $f''(x) = 2$ for $\forall\ x \in (1, 3)$

(b) $f''(x) = f'(x) = 5$ for some $x \in (2, 3)$

(c) $f''(x) = 3$ for $\forall\ x \in (2, 3)$

(d) $f''(x) = 2$ for some $x \in (1, 3)$

5. The function $f(x) = |ax - b| + c|x|$, $\forall\ x \in (-\infty, \infty)$ where $a > 0, b > 0, c > 0$, assumes its minimum value only at one point if

(a) $a \neq b$ (b) $a \neq c$ (c) $b \neq c$ (d) $a = b = 0$

6. The value of c in Lagrange's Theorem for the function $f(x) = \begin{cases} x\cos\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, in the interval $[-1, 1]$ is

(a) 0 (b) $\frac{1}{2}$

(c) $-\frac{1}{2}$ (d) none of these

7. $\lim\limits_{x\to 0}\left(\frac{\sin hx}{x}\right)^{\frac{1}{x^2}} =$

(a) $e^{1/2}$ (b) 1 (c) $e^{1/6}$ (d) $e^{1/3}$

8. The function f defined by

$$f(x) = \begin{cases} \frac{1}{2}, & \text{if } x \text{ is rational} \\ \frac{1}{3}, & \text{if } x \text{ is irrational} \end{cases}$$ is

(a) discontinuous for all x

(b) continuous at $x = 2$

(c) continuous at $x = \frac{1}{2}$

(d) continuous at $x = 3$

9. If a function $f: [-2a, 2a] \to R$ is an odd function such that $f(2a - x) = f(x)$, $\forall\ x \in [a, 2a]$ and left hand derivative at $x = a$ is 0, then find left hand derivative at $x = -a$.

(a) 0 (b) -1 (c) $\frac{1}{2}$ (d) a

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10. $\frac{d^2x}{dy^2} =$

(a) $\left(\frac{d^2y}{dx^2}\right)^{-1}$ (b) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$

(c) $\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^{-2}$ (d) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$

11. If the function $f(x) = x^3 + 3(a-7)x^2 + 3(a^2-9)x - 1$ has a point of maximum at positive values of x, then

(a) $a \in \left(-\infty, \frac{29}{7}\right)$

(b) $a \in (-\infty, 7)$

(c) $a \in (-\infty, -3) \cup \left(3, \frac{29}{7}\right)$

(d) $a \in (3, \infty) \cup (-\infty, -3)$

12. If $\sqrt{x^2+y^2} = a\,e^{\tan^{-1}\left(\frac{y}{x}\right)}$, $a > 0, y(0) > 0$, then $y''(0) =$

(a) $\frac{a}{2}e^{\pi/2}$ (b) $ae^{-\pi/2}$

(c) $\frac{-2}{a}e^{-\pi/2}$ (d) $\frac{a}{2}e^{-\pi/2}$

13. $\int \frac{x^5}{1-x^9}\,dx$ is equal to

(a) $\frac{1}{9}\log(x^3-1) - \frac{1}{18}\log(x^6+x^3+1) - \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{2x^3+1}{\sqrt{3}}\right) + c$

(b) $-\frac{1}{9}\log(x^3-1) + \frac{1}{18}\log(x^6+x^3+1) - \frac{1}{3\sqrt{3}}\tan^{-1}\left(\frac{2x^3+1}{\sqrt{3}}\right) + c$

(c) $-\frac{1}{9}\log(x^3+1) + \frac{1}{18}\log(x^6-x^3+1) - \frac{1}{3\sqrt{3}}\tan^{-1}\left(\frac{2x^3+1}{\sqrt{3}}\right) + c$

(d) none of these

14. $\int_0^1 \frac{\log x}{\sqrt{1-x^2}}\,dx$ is equal to

(a) $-\frac{\pi}{2}\log 2$ (b) $-\frac{\pi}{4}\log 2$

(c) $-\frac{\pi}{8}\log 2$ (d) $\frac{\pi}{4}\log 2$

15. $\int \sqrt{(x-3)}\{\sin^{-1}(\ln x) + \cos^{-1}(\ln x)\}dx$ is equal to

(a) $\frac{\pi}{3}(x-3)^{3/2} + c$ (b) 0

(c) does not exist (d) none of these

16. $\int |x| \ln |x|\,dx$ is equal to $(x \neq 0)$

(a) $\frac{x^2}{2}\ln|x| - \frac{x^2}{4} + c$

(b) $\frac{1}{2}x|x|\ln x + \frac{1}{4}x|x| + c$

(c) $-\frac{x^2}{2}\ln|x| + \frac{x^2}{4} + c$

(d) $\frac{1}{2}x|x|\ln|x| - \frac{1}{4}x|x| + c$

17. Let $f(x) = \min\cdot\left\{x+1, \sqrt{(1-x)}\right\}$, then area bounded by $f(x)$ and x-axis in $[-1, 1]$ is

(a) $\frac{1}{6}$ sq. unit (b) $\frac{5}{6}$ sq. unit

(c) $\frac{7}{6}$ sq. units (d) $\frac{11}{6}$ sq. units

18. The differential equation representing the family of the curves $y^2 = 2c(x + \sqrt{c})$, where c is a positive parameter, is

(a) order 1, degree 3 (b) order 2, degree 2

(c) order 3, degree 3 (d) order 4, degree 4

19. A and B throw alternatively with a pair of dice. A wins if he throws a sum 6 before B throws 7 and B wins if he throws a 7 before A throws 6. If A starts the game, his chance of winning is

(a) $\frac{30}{61}$ (b) $\frac{31}{61}$

(c) $\frac{15}{61}$ (d) $\frac{60}{61}$

20. From a group of n persons arranged in a circle, 3 persons are selected at random. If the probability that no two adjacent persons are selected is $\frac{2}{7}$, then $n =$

(a) 6 (b) 7 (c) 8 (d) 9

21. A box contains 5 pairs of shoes. If 6 shoes are selected at random, the probability that exactly one pair of shoes is obtained is

(a) $\frac{4}{21}$ (b) $\frac{2}{21}$ (c) $\frac{8}{21}$ (d) $\frac{16}{21}$

22. An unbiased coin is tossed 12 times .The probability that at least 7 consecutive heads show up is

(a) $\frac{1}{128}$ (b) $\frac{1}{64}$ (c) $\frac{9}{256}$ (d) $\frac{7}{256}$

23. Six fair dice are thrown independently .The probability that there are exactly 2 different pairs (A pair is an ordered combination like 2, 2, 1, 3, 5, 6) is

(a) $\frac{5}{72}$ (b) $\frac{25}{72}$ (c) $\frac{125}{144}$ (d) $\frac{5}{36}$

24. Two integers x and y are chosen from the set $\{0, 1, 2, 3, \ldots\ldots, 2n\}$, with replacement, the probability that $|x - y| \le n$, $(n \in N)$ is

(a) $\frac{3n^2+3n+1}{(2n+1)^2}$ (b) $\frac{3n^2+3n}{(2n+1)^2}$

(c) $\frac{3n^2+1}{(2n+1)^2}$ (d) $\frac{n^2+n+1}{(2n+1)^2}$

25. If Δ is the area of a triangle whose sides are a, b, c and $\Delta \le \frac{t}{4}$, then $t =$

(a) $\sqrt{(a+b-c)abc}$ (b) $\sqrt{a+b+c}$

(c) $\sqrt{abc}$ (d) $\sqrt{abc(a+b+c)}$

26. The least value of 'c' for which $\frac{a^2}{\sin x} + \frac{b^2}{1-\sin x} = c$ $(a > b > 0)$ has at least one solution in $\left(0, \frac{\pi}{2}\right)$ is

(a) $(a - b)^2$ (b) $a^2 + b^2$

(c) $(a + b)^2$ (d) $a^2 - b^2$

27. If in the triangle ABC, $\frac{AB}{AC} = \frac{3}{4}$ and D is a point lying on BC such that $AB \perp AD$ and D divides BC externally in the ratio 3 : 2, then $\angle A =$

(a) 30° (b) 75° (c) 45° (d) 60°

28. Unit vector $\hat{c}$ is inclined at an angle 'θ' to unit vectors $\hat{a}$ and $\hat{b}$, which are perpendicular to each other. For $\hat{c} = m(\hat{a} + \hat{b}) + n(\hat{a} \times \hat{b})$; m, n real, θ lies in

(a) $\left[0, \frac{\pi}{2}\right]$ (b) $\left[\frac{\pi}{2}, \pi\right]$

(c) $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$ (d) $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

29. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three non-coplanar unimodular vectors, each inclined with other at an angle 30°, then volume of tetrahedran whose edges are $\vec{a}, \vec{b}$ and $\vec{c}$ is

(a) $\frac{\sqrt{3\sqrt{3}-5}}{12}$ (b) $\frac{3\sqrt{3}+5}{12}$

(c) $\frac{5\sqrt{2}+3}{12}$ (d) $\frac{\sqrt{5\sqrt{2}-3}}{13}$

30. $\sum_{r=1}^{\infty} \cot^{-1}\left(r^2 + \frac{3}{4}\right) =$ _____

(a) $\tan^{-1}(2)$ (b) $\cot^{-1}(2)$

(c) $\tan^{-1}(1)$ (d) $\frac{\pi}{2} + \tan^{-1}\left(\frac{1}{2}\right)$

31. The areas of a circle and a regular polygon of n sides and of equal perimeter are in the ratio

(a) $\cot\frac{\pi}{n} : \frac{\pi}{n}$ (b) $\cos\frac{\pi}{n} : \frac{\pi}{n}$

(c) $\tan\frac{\pi}{n} : \frac{\pi}{n}$ (d) none of these

32. The altitudes from the vertices A, B, C of an acute angled ΔABC to the opposite sides meet the circumcircle at D, E, F respectively, then $\frac{EF}{BC} =$

(a) $\sin A$ (b) $\cos A$

(c) $2\sin A$ (d) $2\cos A$

33. In ΔABC; $3\sin A + 4\cos B = 6$ and $4\sin B + 3\cos A = 1$, then $\angle C =$

(a) 30° (b) 150°

(c) 30° or 150° (d) none of these

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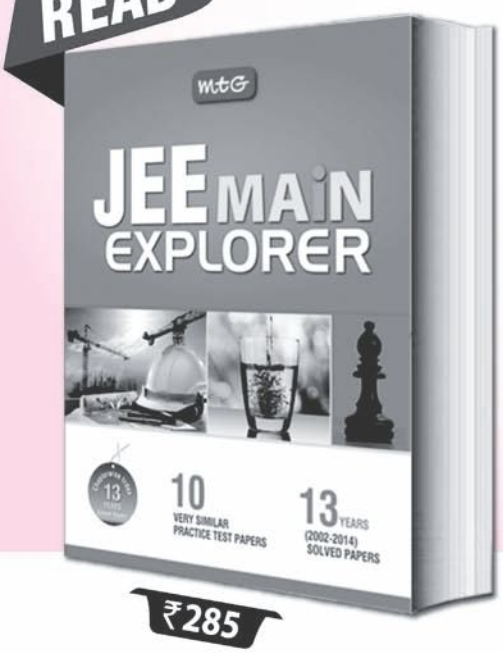

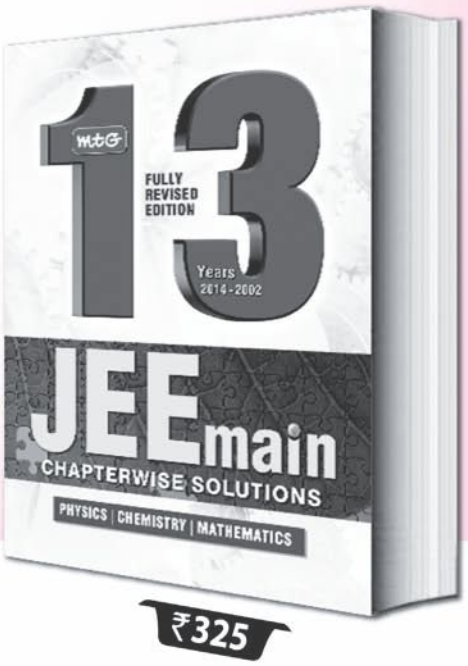

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34. The smallest positive integer 'n' such that

$$\frac{1}{\sin 45^\circ \sin 46^\circ} + \frac{1}{\sin 47^\circ \sin 48^\circ} + \ldots\ldots + \frac{1}{\sin 133^\circ \sin 134^\circ} = \frac{1}{\sin n^\circ} \text{ is}$$

(a) 2 (b) 1 (c) 3 (d) 4

35. In ΔABC, $\sin A + \sin B + \sin C \le 1$, then min $\{A + B, B + C, C + A\}$

(a) $\le 30^\circ$ (b) $< 30^\circ$
(c) $< 60^\circ$ (d) $\le 60^\circ$

36. Let $\vec{a}, \vec{b}, \vec{c}$ be three non coplanar vectors and $\vec{r}$ be any arbitrary vector, then the expression $(\vec{a} \times \vec{b}) \times (\vec{r} \times \vec{c}) + (\vec{b} \times \vec{c}) \times (\vec{r} \times \vec{a}) + (\vec{c} \times \vec{a}) \times (\vec{r} \times \vec{b})$ is equal to

(a) $[\vec{a}\ \vec{b}\ \vec{c}]\vec{r}$ (b) $2[\vec{a}\ \vec{b}\ \vec{c}]\vec{r}$
(c) $3[\vec{a}\ \vec{b}\ \vec{c}]\vec{r}$ (d) none of these

37. Angle between $\hat{k}$ and the line of intersection of the planes $\vec{r} \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) = 0$ and $\vec{r} \cdot (3\hat{i} + 3\hat{j} + \hat{k}) = 0$ is equal to

(a) $\cos^{-1}\left(\frac{2}{\sqrt{3}}\right)$ (b) $\cos^{-1}\left(\frac{3}{\sqrt{122}}\right)$

(c) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (d) $\cos^{-1}\left(\frac{7}{\sqrt{122}}\right)$

38. The parabola $y = x^2 - 8x + 15$ cuts the x-axis at P and Q. A circle is drawn through P and Q so that the origin 'O' is outside. Then the length of the tangent to the circle from 'O' is

(a) 15 (b) $\sqrt{8}$ (c) $\sqrt{15}$ (d) 8

39. Let PQ, RS are the tangents of the extremities of a diameter PR of a circle of radius 'r' such that PS, RQ intersect at a point X on the circumference of the circle, then $2r$ is equal to

(a) $\frac{PQ + RS}{2}$ (b) $\sqrt{PQ \cdot RS}$

(c) $\frac{2PQ \cdot RS}{PQ + RS}$ (d) $\sqrt{\frac{PQ^2 + RS^2}{2}}$

40. Consider the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ $(a > 0)$ have a common tangent at the points P and Q respectively. Then the length of PQ is

(a) $5\sqrt{2}a$ (b) $4\sqrt{2}a$
(c) $3\sqrt{2}a$ (d) $2\sqrt{2}a$

SOLUTIONS

1. (a) : Let (x, y, z) be the circumcentre of space triangle ABC then

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad \ldots(i)$$

and $(x - a)^2 + y^2 + z^2 = x^2 + (y - b)^2 + z^2$
$= x^2 + y^2 + (z - c)^2$

From these relations we can easily get

$$x = \frac{a^2 - b^2 + 2by}{2a}, z = \frac{c^2 - b^2 + 2by}{2c}$$

$$y = \frac{b(a^2b^2 + b^2c^2)}{2k^4}$$

where $k^4 = a^2b^2 + b^2c^2 + c^2a^2$

$$\frac{b^3(a^2 + c^2)}{2k^4} = y_0$$

We can write similar expressions for x and z. Now if y coordinate of orthocentre is y' then we know that centroid divide the line segment joining the orthocentre and circumcentre internally in the ratio 2 : 1, we have

$$\frac{b}{3} = \frac{1 \times y' + 2 \times y_0}{3}$$

$$\Rightarrow y' = b - \frac{b^3(a^2 + c^2)}{a^2b^2 + b^2c^2 + c^2a^2} = \frac{ba^2c^2}{k^4}$$

2. (c) : Since A is image of $P(a, b, c)$ in yz plane, A must be the point. $(-a, b, c)$. Similarly B and C be the points $(a, -b, c)$ and $(a, b, -c)$

$\Rightarrow$ Equation of plane ABC

$bcx + acy + abz = abc$

3. (b) : Clearly $f(x)$ is increasing in $[2, 3]$

$f(x)_{max} = f(3) = \log_{10} 30 = 1 + \log_{10} 3$

4. (d) : Let $g(x) = f(x) - x^2$

Using Rolle's Theorem, we get

$g'(x) = 0$ for at least one $x \in (1, 2)$

Also, $g'(x) = 0$ for at least one $x \in (2, 3)$

$\therefore$ $g''(x) = 0$ for at least one $x \in (1, 3)$

$\therefore$ $f''(x) = 2$ for some $x \in (1, 3)$

5. (b) : $f(x) = \begin{cases} b - (a + c)x & , \quad x < 0 \\ b + (c - a)x & , \quad 0 \le x < \frac{b}{a} \\ (a + c)x - b & , \quad x \ge \frac{b}{a} \end{cases}$

6. (d) : $\therefore$ Clearly $f(x)$ is not differentiable at $x = 0$.

7. (c) : Let $l = \underset{x\to 0}{\text{Lt}}\left(\frac{\sin hx}{x}\right)^{\frac{1}{x^2}}$

$\log l = \underset{x\to 0}{\text{Lt}}\frac{1}{x^2}\log\left(\frac{\sin hx}{x}\right)$ by L-Hospitals Rule

$\Rightarrow \quad l = e^{1/6}$

8. (a) : There lie infinitely many rationals as well as infinitely many irrationals between any two rational or irrational.

If x is rational $\therefore\ \forall\ n \in N\ \exists$ an irrational number x_n such that

$$x - \frac{1}{n} < x_n < x + \frac{1}{n} \Rightarrow |x_n - x| < \frac{1}{n}, \forall n \in N$$

$\Rightarrow \underset{n\to\infty}{\text{Lt}} f(x_n) = \frac{1}{3}$, Similarly in case of irrational.

9. (a) : $f'(-a^-) = \underset{h\to 0}{\text{Lt}}\frac{f(-a) - f(-a-h)}{h}$

$$= \underset{h\to 0}{\text{Lt}}\frac{-f(a) + f(a+h)}{h} = \underset{h\to 0}{\text{Lt}}\frac{-f(a) + f(2a-(a-h))}{h}$$

$$= \underset{h\to 0}{\text{Lt}}\frac{-f(a) + f(a-h)}{h} = 0$$

10. (d) : $\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1}$

Differentiating with respect to y.

11. (c) : $f(x) = x^3 + 3(a-7)x^2 + 3(a^2-9)x - 1$

$f'(x) = 3x^2 + 6(a-7)x + 3(a^2-9)$

The roots of $f'(x) = 0$ are positive and distinct which is possible if

(i) $b^2 - 4ac > 0 \Rightarrow 36(a-7)^2 - 4(3)(3)(a^2-9) > 0$

$\Rightarrow \quad a < \frac{29}{7}$

(ii) Product of Roots $> 0 \Rightarrow a^2 - 9 > 0$

(iii) Sum of Roots $> 0 \Rightarrow a - 7 < 0 \Rightarrow a < 7$

$\Rightarrow$ From (i), (ii), (iii) $a \in (-\infty, -3) \cup \left(3, \frac{29}{7}\right)$

12. (c) : Differentiating with respect to x two times, we get

$$\frac{d^2y}{dx^2} = \frac{2(x^2+y^2)}{(x-y)^3}$$

From given eqn. when $x = 0 \Rightarrow y = ae^{\pi/2}$

$$\left[\frac{d^2y}{dx^2}\right]_{\text{at } x=0} = \frac{-2}{y} = \frac{-2}{a}e^{-\pi/2}$$

13. (b) : Putting $x^3 = z$, then $3x^2dx = dz$

$$\therefore \quad I = \frac{1}{3}\int \frac{zdz}{(1-z)(1+z+z^2)}$$

Let $\frac{z}{(1-z)(1+z+z^2)} \equiv \frac{A}{1-z} + \frac{Bz+C}{1+z+z^2}$

$$I = \frac{1}{3}\int\frac{1}{3}.\frac{dz}{1-z} + \frac{1}{3}\int\frac{\frac{1}{3}(z-1)}{1+z+z^2}dz$$

$$= -\frac{1}{9}\log(x^3-1) + \frac{1}{18}\log(x^6+x^3+1)$$

$$-\frac{1}{3\sqrt{3}}\tan^{-1}\left(\frac{2x^3+1}{\sqrt{3}}\right) + c$$

14. (a) : Put $x = \sin\theta$ then $dx = \cos\theta d\theta$

And $x = 0 \Rightarrow \theta = 0;\ x = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$\therefore \quad I = \int_0^{\pi/2}\frac{\log\sin\theta}{\sqrt{1-\sin^2\theta}}\cos\theta d\theta$$

$$= \int_0^{\pi/2}\log\sin\theta d\theta = \int_0^{\pi/2}\log\sin\left(\frac{\pi}{2}-\theta\right)d\theta$$

using $\int_0^a f(x)dx = \int_0^a f(a-x)dx \quad \therefore \quad I = -\frac{\pi}{2}\log 2$

15. (c) : $\because\ \sqrt{x-3}$ is defined only when $x \geq 3$ and $\sin^{-1}(\ln x) + \cos^{-1}(\ln x)$ is defined only when,

$$-1 \leq \ln x \leq 1 \Rightarrow \frac{1}{e} \leq x \leq e$$

Then, $[3,\infty) \cup \left[\frac{1}{e}, e\right] = \phi$

Hence the given integral does not exist.

16. (d) : Case I : If $x > 0$, then $|x| = x$

$$\Rightarrow \int |x|\ln|x|dx = \int x\ln x dx = \frac{x^2}{2}\ln|x| - \frac{x^2}{4} + c$$

Case II : If $x < 0$, then $|x| = -x$

$$\Rightarrow \int |x|\ln|x|dx = -\int x\ln(-x)dx = -\frac{x^2}{2}\ln|x| + \frac{x^2}{4} + c$$

$\Rightarrow$ Combining both cases, we get,

$$\frac{1}{2}x|x|\ln|x| - \frac{1}{4}x|x| + c$$

17. (c) :

$$f(x)=\min\{x+1,\sqrt{(1-x)}\}=\begin{cases}x+1, & -1\le x<0\\ \sqrt{1-x}, & 0<x\le 1\end{cases}$$

$$\therefore \text{ required area}=\left|\int_{-1}^{0}(x+1)dx\right|+\left|\int_{0}^{1}\sqrt{1-x}\,dx\right|$$

$$=\frac{7}{6}\text{ sq. units}$$

18. (a) : $\because \quad y^2=2cx+2c^{3/2}$

$\Rightarrow 4y(y')^3+4xyy'-y^2-4x^2(y')^2=0$

$\Rightarrow$ order =1, degree = 3

19. (a) : A's chance of winning in a throw $=\frac{5}{36}$,

B's chance of winning in a throw $=\frac{1}{6}$

A's chance of losing in a throw $=\frac{31}{36}$,

B's chance of losing in a throw $=\frac{5}{6}$

A can winning the game

$$=\frac{5}{36}+\frac{31}{36}\times\frac{5}{6}\times\frac{5}{36}+\frac{31}{36}\times\frac{5}{6}\times\frac{31}{36}\times\frac{5}{6}\times\frac{5}{36}+.....$$

$$=\frac{5}{36}\left[1+\frac{155}{216}+\left(\frac{155}{216}\right)^2+.....\right]=\frac{30}{61}$$

20. (c) : P(no two adjacent persons are selected)

$$=\frac{n(n-4)(n-5)/6}{n(n-1)(n-2)/6}=\frac{(n-4)(n-5)}{(n-1)(n-2)}=\frac{2}{7}$$

$\Rightarrow 5n^2-57n+136=0 \Rightarrow \quad n=8$

21. (c) : From 10 shoes, 6 can be selected in ${}^{10}C_6$ ways and if there is to be one pair among them it can be selected in ${}^5C_1$ ways. Other 4 shoes can be selected in ${}^4C_4(2^4)=16$ ways.

Hence probability $=\frac{5\times 16}{{}^{10}C_6}=\frac{8}{21}$

22. (d) : The sequence of consecutive heads may starts with 1st toss or 2nd toss or 3rd toss ..... or at 6th toss. In any case ,if it starts with r^{th} throw, the first $(r-2)$ throws may be head or tail but $(r-1)^{st}$ throw must be tail, after which again a head or tail can show up.

$$\therefore \text{ Probability}=\frac{1}{2^7}+\frac{1}{2}\cdot\frac{1}{2^7}+.....+\frac{1}{2}\cdot\frac{1}{2^7}$$

$$=\frac{1}{2^7}\left[1+\frac{5}{2}\right]=\frac{7}{2^8}$$

23. (b) : Total no. of outcomes $=6^6$. Number of ways of choosing 4 other different numbers is ${}^6C_2$ and choosing 2 out of remaining 4 can be done in ${}^4C_2$ ways. Also number of ways of arranging 6 numbers of which 2 are alike and 2 are alike is $\frac{6!}{2!2!}$.

$$\text{Required probability}=\frac{{}^6C_2\times{}^4C_2\times\frac{6!}{2!2!}}{6^6}=\frac{25}{72}$$

24. (a) : x and y can be any one of $(2n+1)$ numbers given $|x-y|\le n \Rightarrow x-n\le y\le x+n$

Hence number of possibilities of y, for $x=0,1,2,3,$ ..... $n-1, n, n+1$ ..... $2n$ are

$n+1, n+2, n+3,$ ....., $2n, 2n+1, 2n, 2n-1,$ ..... $n+1$ respectively.

$$\therefore \text{ Probability}=\frac{2(\overline{n+1}+\overline{n+2}+.....+\overline{2n})+2n+1}{(2n+1)^2}$$

$$=\frac{3n^2+3n+1}{(2n+1)^2}$$

25. (d) : $\frac{(s-a)+(s-b)}{2}\ge\sqrt{(s-a)(s-b)}$

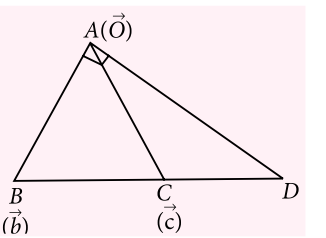
$\therefore \quad \frac{c}{2}\ge\sqrt{(s-a)(s-b)}$ Similarly others

$\therefore \quad \frac{abc}{8}\ge\frac{\Delta^2}{s} \Rightarrow \Delta\le\frac{\sqrt{(a+b+c)abc}}{4}$

26. (c) : Solve $\frac{dc}{dx}=0$ and get $\sin x=\frac{a}{a+b}$ as $\frac{a}{a-b}>1$ is not possible.

Then find minimum value of c.

27. (d) :



$\therefore \quad$ P.V. of $D=3\vec{c}-2\vec{b}$

$\overrightarrow{AB}\cdot\overrightarrow{AD}=0 \Rightarrow \cos A=\frac{2|\vec{b}|}{3|\vec{c}|}=\frac{1}{2} \quad \therefore \quad A=60°$

28. (c) : $\hat{c}\cdot\hat{a}=\cos\theta \Rightarrow m=\cos\theta$

Also $\hat{c}\cdot\hat{c}=1 \Rightarrow 2m^2+n^2=1$

$\Rightarrow n^2 = 1 - 2\cos^2\theta \geq 0 \Rightarrow \frac{-1}{\sqrt{2}} \leq \cos\theta \leq \frac{1}{\sqrt{2}}$

29. (a) : $V^2=\frac{1}{36}\begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c}\\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c}\\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c}\end{vmatrix}$

30. (a) :

$$\cot^{-1}\left(r^2+\frac{3}{4}\right)=\tan^{-1}\left(r+\frac{1}{2}\right)-\tan^{-1}\left(r-\frac{1}{2}\right)$$

$$\therefore \sum_{r=1}^{\infty}\cot^{-1}\left(r^2+\frac{3}{4}\right)=\frac{\pi}{2}-\tan^{-1}\left(\frac{1}{2}\right)$$

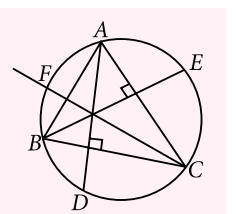
$$=\cot^{-1}\left(\frac{1}{2}\right)=\tan^{-1}(2)$$

31. (c) : 's' – side of polygon, r – radius of the circle.

$2\pi r = ns$.

$$\therefore \text{ Required ratio}=\frac{\pi r^2}{\frac{ns^2}{4}\cot\frac{\pi}{n}}=\tan\frac{\pi}{n}:\frac{\pi}{n}$$

32. (d) :



$\angle ADE = \angle ABE = 90° - A$

$\angle ADF = \angle ACF = 90° - A$

$\therefore \quad \angle D = 180° - 2A$

$$\frac{EF}{BC}=\frac{2R\sin D}{2R\sin A}=2\cos A$$

33. (a) : Squaring and adding the given equations, we get

$24\sin(A + B) = 12 \Rightarrow \sin C = \frac{1}{2}$

$\Rightarrow$ either $C = 30°$ or $150°$

But if $C = 150°$; $A < 30°$ so that

$3\sin A + 4\cos B < \frac{3}{2} + 4 < 6$, a contradiction.

34. (b) : $\sin(1°) = \sin((x + 1)° - x°)$

$$\therefore \frac{\sin(1°)}{\sin x°\sin(x+1)°}=\cot x°-\cot(x+1)°$$

Using this, we can arrive $n = 1$.

35. (b) : Without loss of generality assume $A \geq B \geq C$

We have $b + c > a$

$\Rightarrow \sin B + \sin C > \sin A$

$\Rightarrow \sin A + \sin B + \sin C = 2\sin A$

$\therefore$ It follows $\sin A < \frac{1}{2}$ and $A \geq \frac{A+B+C}{3} = 60°$

which gives $A > 150°$; *i.e.* $B + C < 30°$

36. (b) : Given expression can be written as

$$3\left[\vec{a}\,\vec{b}\,\vec{c}\right]\vec{r}-\left\{\left[\vec{a}\,\vec{b}\,\vec{r}\right]\vec{c}+\left[\vec{b}\,\vec{c}\,\vec{r}\right]\vec{a}+\left[\vec{c}\,\vec{a}\,\vec{r}\right]\vec{b}\right\}$$

$$=3\left[\vec{a}\,\vec{b}\,\vec{c}\right]\vec{r}-\left[\vec{a}\,\vec{b}\,\vec{c}\right]\vec{r}=2\left[\vec{a}\,\vec{b}\,\vec{c}\right]\vec{r}$$

37. (b) : Line of intersection of the given planes will be parallel to $(\hat{i}+2\hat{j}+3\hat{k})\times(3\hat{i}+3\hat{j}+\hat{k})$

38. (c) : The points P and Q are (3, 0), (5, 0)

$\therefore$ The equation of the circle is

$x^2 + y^2 - 8x + 2fy + 15 = 0$;

Length of tangent from origin $= \sqrt{c} = \sqrt{15}$

39. (b) : $\frac{PQ}{PR}=\tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta$; $\frac{RS}{PR}=\tan\theta$

$$\therefore \frac{PQ\cdot RS}{PR^2}=1 \Rightarrow PR^2=PQ\cdot RS$$

$$\therefore 2r=\sqrt{PQ\cdot RS}$$

40. (c) : Points of contacts are ends of latusrectum *i.e.*, $P(a, -2a), Q(-2a, a) \Rightarrow PQ = 3\sqrt{2}\,a$

IMPORTANT EXAMINATION DATES 2015

4 April	JEE Main (Offline)	29-30 April	Karnataka CET
10 – 11 April	JEE Main (Online)	3 May	AIPMT
8 – 19 April	VITEEE	10 May	COMED K
18-19 April	WB JEE	16 May	UPSEAT
19 April	MGIMS	24 May	JEE Advanced
20-21April	Kerala PET	1 June	AIIMS
22-23 April	Kerala PMT	7 June	JIPMER

YOU ASKED WE ANSWERED

Do you have a question that you just can't get answered?

Use the vast expertise of our mtg team to get to the bottom of the question. From the serious to the silly, the controversial to the trivial, the team will tackle the questions, easy and tough.

The best questions and their solutions will be printed in this column each month.

1. If $f(x)=\sin^2 x+\sin^2\left(x+\frac{\pi}{3}\right)+\cos x\cos\left(x+\frac{\pi}{3}\right)$ and $g\left(\frac{5}{4}\right)=1$, then find the value of $(gof)\left(\frac{\pi}{8}\right)$.

- Richa Kaushik (Hyderabad)

Ans. We have,

$$f(x)=\sin^2 x+\sin^2\left(x+\frac{\pi}{3}\right)+\cos x\cos\left(x+\frac{\pi}{3}\right)$$

$$=\frac{1}{2}\left[2\sin^2 x+1-\cos\left(2x+\frac{2\pi}{3}\right)+\cos\left(2x+\frac{\pi}{3}\right)+\cos\frac{\pi}{3}\right]$$

$$=\frac{1}{2}\left[2\sin^2 x-\cos\left(2x+\frac{2\pi}{3}\right)+\cos\left(2x+\frac{\pi}{3}\right)+\frac{3}{2}\right]$$

$$=\frac{1}{2}\left[2\sin^2 x+\cos 2x+\frac{3}{2}\right]$$

$$=\frac{1}{2}\left(1+\frac{3}{2}\right)=\frac{5}{4}$$

Therefore $f(x) = 5/4$ for all x. Hence

$$(gof)\left(\frac{\pi}{8}\right)=g\left(\frac{5}{4}\right)=1$$

2. A letter is to come from either LONDON or CLIFTON. The postal mark on the letter legibly shows consecutive letters "ON". What is the probability that the letter has come from LONDON ?

- Nihal Pandey (Bihar)

Ans. Let the event be defined as

E_1 : Letter coming from LONDON.

E_2 : Letter coming from CLIFTON.

E : Two consecutive letters ON.

The word LONDON contains 5 types of consecutive letters (LO, ON, ND, DO, ON) of which there are two ON's. The word CLIFTON contains 6 types of consecutive letters (CL, LI, IF, FT, TO, ON) of which there is one "ON". Now

$$P(E_1)=\frac{1}{2}=P(E_2)$$

$$P(E\mid E_1)=\frac{2}{5}\text{ and }P(E\mid E_2)=\frac{1}{6}$$

By Bayes' theorem,

$$P(E_1\mid E)=\frac{\frac{1}{2}\times\frac{2}{5}}{\frac{1}{2}\times\frac{2}{5}+\frac{1}{2}\times\frac{1}{6}}=\frac{12}{17}$$

3. Evaluate : $\int_{-1}^{1}\frac{x^3+|x|+1}{x^2+2|x|+1}dx$

- Niyati Shukla (U.P.)

Ans. $\int_{-1}^{1}\frac{x^3+|x|+1}{x^2+2|x|+1}dx=\int_{-1}^{1}\frac{x^3+(|x|+1)}{(|x|+1)^2}dx$

$(\because |x|^2=x^2)$

$$=\int_{-1}^{1}\frac{x^3}{(|x|+1)^2}dx+\int_{-1}^{1}\frac{1+|x|}{(|x|+1)^2}dx$$

$$=0+\int_{-1}^{1}\frac{1}{1+|x|}dx$$

$\left(\because \frac{x^3}{(|x|+1)^2}\text{ is an odd function}\right)$

$$=2\int_{0}^{1}\frac{1}{1+|x|}dx$$

$$=2\int_{0}^{1}\frac{1}{1+x}dx\quad\left(\because \frac{1}{1+|x|}\text{ is an even function}\right)$$

$$=2[\log|1+x|]_0^1=2(\log 2-\log 1)=2\log 2$$

PRACTICE PAPER 2015

ISI
(Indian Statistical Institute)

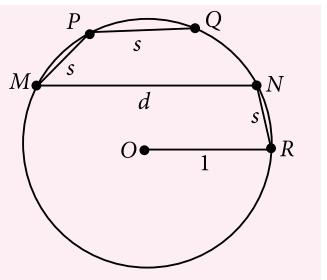
Exam on 10th May

* ALOK KUMAR, B.Tech, IIT Kanpur

PART-A

Multiple Choice Test

1. In the unit circle shown in the figure to the right, chords PQ and MN are parallel to the unit radius OR of the circle with centre at O. Chords MP, PQ and NR are each s units long and chord MN is d units long. Of the three equations

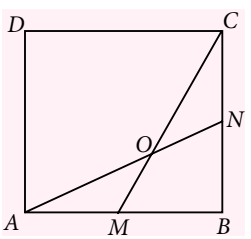


I. $d - s = 1$, II. $ds = 1$, III. $d^2 - s^2 = \sqrt{5}$

those which are necessarily true are

(a) I only (b) II only
(c) III only (d) I and II only
(e) I, II and III

2. In the adjoining figure AB and BC are adjacent sides of square $ABCD$; M is the mid-point of AB; N is the mid-point of BC; and AN and CM intersect at O. The ratio of the area of $AOCD$ to the area of $ABCD$ is



(a) 5/6 (b) 3/4 (c) 2/3 (d) $\sqrt{3}/2$
(e) $\dfrac{(\sqrt{3}-1)}{2}$

3. A woman, her brother, her son and her daughter are chess players (all relations by birth). The worst player's twin (who is one of the four players) and the best player are of opposite sex. The worst player and the best player are of the same age. Who is the worst player?

(a) The woman (b) Her son
(c) Her brother (d) Her daughter
(e) No solution is consistent with the given information.

4. Given an equilateral triangle with side of length s, consider the locus of all points P in the plane of the triangle such that the sum of the squares of the distances from P to the vertices of the triangle is a fixed number a. This locus

(a) is a circle if $a > s^2$
(b) contains only three points if $a = 2s^2$ and is a circle if $a > 2s^2$
(c) is a circle with positive radius only if $s^2 < a < 2s^2$
(d) contains only a finite number of points for any value of a
(e) none of these

5. What is the smallest integer larger than $(\sqrt{3}+\sqrt{2})^6$?

(a) 972 (b) 971 (c) 970 (d) 969
(e) 968

* Alok Kumar is a winner of INDIAN NATIONAL MATHEMATICS OLYMPIAD (INMO-91). He trains IIT and Olympiad aspirants.

6. How many distinct ordered triples (x, y, z) satisfy the equations

$x + 2y + 4z = 12$; $xy + 4yz + 2xz = 22$

$xyz = 6$?

(a) None (b) 1 (c) 2 (d) 4

(e) 6

7. For how many paths consisting of a sequence of horizontal and/or vertical line segments, with each segment connecting a pair of adjacent letters in the diagram below, is the word CONTEST spelled out as the path is traversed from beginning to end ?

(a) 63 (b) 128 (c) 129 (d) 255

(e) None of these

C

COC

CONOC

CONTNOC

CONTETNOC

CONTESETNOC

CONTESTSETNOC

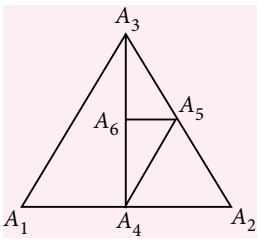
8. Let $g(x) = x^5 + x^4 + x^3 + x^2 + x + 1$. What is the remainder when the polynomial $g(x^{12})$ is divided by the polynomial $g(x)$?

(a) 6 (b) $5 - x$

(c) $4 - x + x^2$ (d) $3 - x + x^2 - x^3$

(e) $2 - x + x^2 - x^3 + x^4$

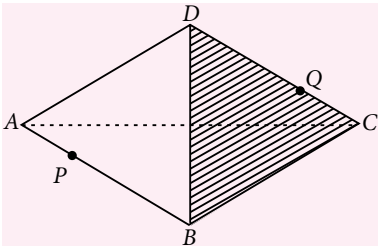
9. If $\Delta A_1A_2A_3$ is equilateral and A_{n+3} is the mid-point of line segment A_nA_{n+1} for all positive integers n, then the measure of $\angle A_{44}A_{45}A_{43}$ equals



(a) 30° (b) 45° (c) 60° (d) 90°

(e) 120°

10. The edges of a regular tetrahedron with vertices A, B, C and D each have length one. Find the least possible distance between a pair of points P and Q, where P is on edge AB and Q is on edge CD.



(a) $\frac{1}{2}$ (b) $\frac{3}{4}$ (c) $\frac{\sqrt{2}}{2}$ (d) $\frac{\sqrt{3}}{2}$

(e) $\frac{\sqrt{3}}{3}$

11. A bug (of negligible size) starts at the origin on the coordinate plane. First it moves 1 unit right to (1, 0). Then it makes a 90° turn counterclockwise and travels 1/2 a unit to (1, 1/2). If it continues in this fashion, each time making a 90° turn counter clockwise and travelling half as far as in the previous move, to which of the following points will it come closest ?

(a) $\left(\frac{2}{3},\frac{2}{3}\right)$ (b) $\left(\frac{4}{5},\frac{2}{5}\right)$ (c) $\left(\frac{2}{3},\frac{4}{5}\right)$ (d) $\left(\frac{2}{3},\frac{1}{3}\right)$

(e) $\left(\frac{2}{5},\frac{4}{5}\right)$

12. If the function f defined by $f(x)=\frac{cx}{2x+3}$, $x\neq-\frac{3}{2}$, c a constant, satisfies $f(f(x)) = x$ for all real number x except $-3/2$, then c is

(a) -3 (b) $-3/2$ (c) $3/2$ (d) 3

(e) Not uniquely determined by the given information

13. The polynomial $x^{2n} + 1 + (x + 1)^{2n}$ is not divisible by $x^2 + x + 1$ if n equals

(a) 17 (b) 20 (c) 21 (d) 64

(e) 65

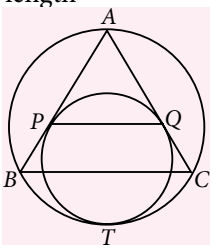
14. The number of real solutions to the equation $\frac{x}{100}=\sin x$ is

(a) 61 (b) 62 (c) 63 (d) 64

(e) 65

15. Equilateral ΔABC is inscribed in a circle. A second circle is tangent internally to the circumcircle at T and tangent to sides AB and AC

at points P and Q. If side BC has length 12, then segment PQ has length

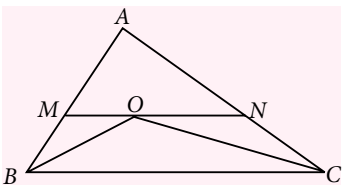


(a) 6 (b) $6\sqrt{3}$ (c) 8 (d) $8\sqrt{3}$
(e) 9

16. If the base 8 representation of a perfect square is $ab3c$, where $a \neq 0$, then c is
(a) 0 (b) 1 (c) 3 (d) 4
(e) Not uniquely determined

17. A vertical line divides the triangle with vertices (0, 0), (1, 1) and (9, 1) in the xy-plane into two regions of equal area. The equation of the line is $x =$
(a) 2.5 (b) 3.0 (c) 3.5 (d) 4.0
(e) 4.5

18. In the adjoining diagram, BO bisects $\angle CBA$, CO bisects $\angle ACB$, and MN is parallel to BC. If $AB = 12$, $BC = 24$, and $AC = 18$, then the perimeter of ΔAMN is



(a) 30 (b) 33 (c) 36 (d) 39
(e) 42

PART-B

Short - Answer Type Test

19. Find the real roots of the equation

$$x^2+2ax+\frac{1}{16}=-a+\sqrt{a^2+x-\frac{1}{16}}, \text{where } 0<a<\frac{1}{4}.$$

20. Find the real points (x, y) satisfying
$3x^2 + 3y^2 - 4xy + 10x - 10y + 10 = 0$.

21. In an acute-angled triangle ABC, $\angle A = 30°$, O is the orthocentre and M is the mid-point of BC. On the line OM, T is the point such that $OM = MT$. Show that $AT = 2BC$.

22. Let G be the centroid of the triangle ABC in which the angle at C is obtuse and let AD and CF be medians from A and C respectively onto the sides BC and AB. If the four points B, D, G and F are concyclic, show that $\frac{AC}{BC} > \sqrt{2}$. If, further, P is a point on the line BG extended such that $AGCP$ is a parallelogram, show that the triangles ABC and GAP are similar.

23. If $11+11\sqrt{11a^2+1}$ is an odd integer, where a is a rational number, prove that a is a perfect square.

24. Show that there is a natural number n such that $n!$ when written in decimal notation (that is, in base 10) ends exactly in 1993 zeroes.

25. In a triangle ABC, $\angle A$ is twice $\angle B$. Show that $a^2 = b(b + c)$. (In fact, the converse is also true. Prove it.)

26. Suppose P is an interior point of a triangle ABC and AP, BP, CP meet the opposite sides BC, CA, AB in D, E, F respectively. Show that $\frac{AF}{FB}+\frac{AE}{EC}=\frac{AP}{PD}$.

27. There are two urns each containing an arbitrary number of balls. Both are non-empty to begin with. We are allowed two types of operations:
(i) Remove an equal number of balls simultaneously from both urns;
(ii) Double the number of balls in any one of them.
Show that after performing these operations finitely many times, both the urns can be made empty.

28. Find the number of quadratic polynomials, $ax^2 + bx + c$, which satisfy the following conditions:
(i) a, b, c are distinct;
(ii) $a, b, c \in \{1, 2, 3, ..., 1999\}$ and
(iii) $x + 1$ divides $ax^2 + bx + c$.

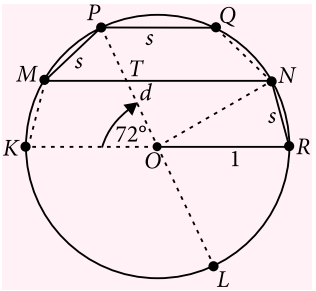
29. At each of the eight corners of a cube write + 1 or – 1 arbitrarily. Then, on each of the six faces of the cube write the product of the numbers written at the four corners of that face. Add all the fourteen numbers so written down. Is it possible to arrange the numbers + 1 and –1 at the corners initially so that this final sum is zero ?

SOLUTIONS

1. (e) : We shall show that I, II and III are all true by giving geometric arguments for I and II and then showing algebraically that III is a consequence of I and II.

In the adjoining figure, chords QN and KM have length s. The five equal chords of length s in the semicircle with diameter KR subtend central angles of $180°/5 = 36°$ each. The five isosceles triangles, each with base s and opposite vertex at the centre O, have base angles of measure

$$\frac{1}{2}(180° - 36°) = 72°.$$



Now rotate the entire configuration clockwise through 72° about O. Then chord PQ, parallel to diameter KR, goes into chord NR, parallel to diameter PL. Denote by T the intersection of MN and PL. In parallelogram $ORNT$, $TN = OR = 1$, and $TO = NR = s$. We saw that $\angle MPO = 72°$; now $\angle MTP = 72°$ also, because $MT \parallel KO$. So ΔPMT is isosceles with $MT = MP = s$.

Therefore $d = MT + TN = s + 1$, so, $d - s = 1$.

The segments PT, TL and MT, TN of intersecting chords MN and PL satisfy $(PT)(TL) = (MT)(TN)$. Since $PT = OP - OT = 1 - s$, and $TL = OL + OT = 1 + s$, this equation takes the form

$$(1 - s)(1 + s) = s \cdot 1 \quad \text{or} \quad 1 - s^2 = s$$

Multiplying equation $s^2 + s = 1$ is equivalent to $s^2 + s - 1 = 0$, whose positive root is $s = \frac{1}{2}(\sqrt{5} - 1)$. Therefore

$$d = s + 1 = \frac{\sqrt{5}}{2} + \frac{1}{2}, \text{ and}$$

III. $d^2 - s^2 = (d + s)(d - s) = d + s = \sqrt{5}$.

Each chord of length s subtends an angle 36° at the centre O, and MN of length d subtends an angle $3 \cdot 36° = 108°$. Therefore,

$s = 2\sin 18°$ and $d = 2\sin 54°$

It follows from this and trigonometric identities that

$d = 2\sin 54° = 2\cos 36° = 2(1 - 2\sin^2 18°) = 2 - s^2$

and

$s = 2\sin 18° = 2\cos 72° = 2(\cos^2 36° - 1) = d^2 - 2$

Adding $d = 2 - s^2$ and $s = d^2 - 2$, we obtain $d + s = d^2 - s^2 - (d + s)(d - s)$, so that I. $d - s = 1$.

Substituting $s + 1$ for d in $d = 2 - s^2$, we find that $s^2 + s - 1 = 0$, which has one positive root, $s = \frac{1}{2}(\sqrt{5} - 1)$. Then

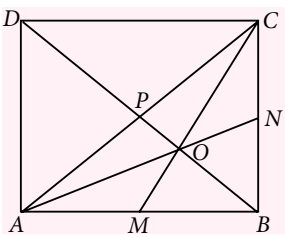
$$d = s + 1 = \frac{\sqrt{5} + 1}{2}, \; ds = 1, \; d + s = \sqrt{5}$$

and $d^2 - s^2 = \sqrt{5}$.

2. (c) : 1st Solution: In the adjoining figure diagonals AC and DB are drawn. Since O is the intersection of the medians of ΔABC, the altitude of ΔAOB from O is 1/3 the altitude of ΔABC from C; *i.e.* 1/3 the side length, s, of the square. Hence

$$\text{area of } \Delta AOB = \frac{1}{3}(\text{area of } \Delta ABC)$$

$$= \frac{1}{3}\left(\frac{1}{2}s^2\right) = \frac{1}{6}s^2$$



Similarly, area of $\Delta COB = \frac{1}{6}s^2$. The area of $AOCD$ is obtained by subtracting the areas of triangles AOB and COB from that of the square, so area of

$$AOCD = s^2 - \frac{1}{3}s^2 - \frac{2}{3}s^2$$

2nd Solution : Introduce coordinates with respect to which AB is the unit interval on the positive x-axis. AD is the unit interval on the positive y-axis. Now,

Area of $AOCD$ = Area of ΔACD + Area of ΔAOC

$$= \frac{1}{2} + \frac{1}{2}\begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ 1 & 1 \end{vmatrix} = \frac{2}{3}$$

The rows of the determinant are the coordinates of O and C. Those of O are $\left(\frac{2}{3}, \frac{1}{3}\right)$ because O is the intersection of medians of ΔABC.

Remark: Since the area of ΔABN is 1/4 of the area of the square, it is clear that the desired ratio r satisfies $\frac{1}{2} < r < \frac{1}{4}$. Only (c) fulfills this condition.

3. (b) : If the son is the worst player, the daughter must be his twin. The best player must then be the brother. This is consistent with the given information, since the brother and the son could be of the same age. The assumption that any of the other players is worst leads to a contradiction:

If the woman is the worst player, her brother must be her twin and her daughter must be the best player. But the woman and her daughter cannot be of the same age.

If the brother is the worst player, the woman must be his twin. The best player is then the son. But the woman and her daughter cannot be of the same age.

If the brother is the worst player, the woman must be his twin. The best player is then the son. But the woman and her son cannot be of the same age, and hence the woman's twin, her brother, cannot be of the same age as the son.

If the daughter is the worst player, the son must be the daughter's twin. The best player must be the woman. But the woman and her daughter cannot be of the same age.

4. (a) : Let point P have coordinates (x, y) in the coordinate system in which the vertices of the equilateral triangle are $(0, 0)$, $(s, 0)$ and $\left(\frac{s}{2}, \frac{s\sqrt{3}}{2}\right)$.

Then P belongs to the locus if and only if

$$a = x^2 + y^2 + \left(x - s\right)^2 + y^2 + \left(x - \frac{s}{2}\right)^2 + \left(y - \frac{s\sqrt{3}}{2}\right)^2,$$

or, equivalently, if and only if

$$a = (3x^2 - 3sx) + (3y^2 - s\sqrt{3}y) + 2s^2$$

$$\frac{a - 2s^2}{3} = \left(x - \frac{s}{2}\right)^2 + \left(y - \frac{s\sqrt{3}}{6}\right)^2 - \frac{s^2}{3},$$

$$\frac{a - s^2}{3} = \left(x - \frac{s}{2}\right)^2 + \left(y - \frac{s\sqrt{3}}{6}\right)^2$$

Thus the locus is the empty set if $a < s^2$; the locus is a single point if $a = s^2$; and the locus is a circle if $a > s^2$.

5. (c) : Instead of trying to compute $(\sqrt{3} + \sqrt{2})^6$ directly, we compute something slightly larger and easier to compute, because many terms cancel; namely we compute

$$(\sqrt{3} + \sqrt{2})^6 + (\sqrt{3} - \sqrt{2})^6.$$

When $(a + b)^{2k}$ and $(a - b)^{2k}$ are expanded by the binomial theorem, their even-powered terms are identical, and their odd-powered terms differ only in sign; so their sum is

$$2\left[a^{2k} + \binom{2k}{2} a^{2k-2}b^2 + \ldots + b^{2k}\right].$$

This principle, applied with $a = \sqrt{3}, b = \sqrt{2}$, $k = 3$ yields

$$(\sqrt{3} + \sqrt{2})^6 + (\sqrt{3} - \sqrt{2})^6 = 2[27 + 15(18 + 12) + 8]$$
$$= 970$$

Since $0 < \sqrt{3} - \sqrt{2} < 1$, 970 is the smallest integer larger than $(\sqrt{3} + \sqrt{2})^6$.

6. (e) : We observe that we can find a system of symmetric equations by the change of variables

(1) $x = 2u, y = v, z = \frac{1}{2}w$.

This substitution yields the transformed system

(2) $u + v + w = 6, uv + vw + uw = 11, uvw = 6$.

Consider the polynomial

$p(t) = (t - u)(t - v)(t - w)$,

where (u, v, w) is a solution of the system (2). Then

(3) $p(t) = t^3 - 6t^2 + 11t - 6$,

and u, v, w are the solutions of $p(t) = 0$. Conversely, if the roots of $p(t) = 0$ are listed as a triple in any order, this triple is a solution to system (2).

If it is not hard to see that $p(t) = 0$ has three distinct solutions; in fact,

$p(t) = (t - 1)(t - 2)(t - 3)$. So the triple (1, 2, 3) and each of its permutations satisfies the system (2). Since the change of variables (1) is one-to-one, the original system has 6 distinct solutions (x, y, z): (2, 3, 1), $\left(2, 2, \frac{3}{2}\right), \left(4, 1, \frac{3}{2}\right), \left(4, 3, \frac{1}{2}\right)$, (6, 1, 1) or $\left(6, 2, \frac{1}{2}\right)$.

7. (e) : All admissible paths end at the centre "T" in the bottom row of the diagram. Our count is easier if we go from the end to the beginning of each path; that is, if we spell TSETNOC, starting at the bottom centre and traversing sequences of horizontally and/or vertically upward directed segments. The count becomes easier still if we take advantage of the symmetry of the figure and distinguish those paths whose horizontal segments are directed to the lift (see figure) from those whose horizontal segments are directed to the right. These two sets have the central vertical column in common and contain an equal number of paths. Starting at the bottom corner "T" in our figure, we have at each stage of the spelling the two choices of taking the next letter from above or from the left. Since there are 6 steps, this leads 10 2^6 paths in this configuration. We get 2^6 paths also in the symmetric configuration. Since we have counted the central column twice, there are altogether $2 \cdot 2^6 - 1 = 127$ distinct paths.

C
CO
CON
CONT
CONTE
CONTES
CONTEST

8. (a) : 1st Solution: We shall use the identity

$$(x-1)(x^n + x^{n-1} + ... + x + 1) = x^{n+1} - 1$$

Thus, for example, $(x-1)g(x) = x^6 - 1$. By definition of the function g, we have

$$g(x^{12}) = (x^{12})^5 + (x^{12})^4 + (x^{12})^3 + (x^{12})^2 + x^{12} + 1$$
$$= (x^6)^{10} + (x^6)^8 + (x^6)^6 + (x^6)^4 + (x^6)^2 + 1$$

Subtracting 1 from each term on the right yields the equation

$$g(x^{12}) - 6 = [(x^6)^{10} - 1] + [(x^6)^8 - 1] + ...$$
$$... + [(x^6)^2 - 1]$$

Each expression on the right is divisible by $x^6 - 1$. We may therefore write

$$g(x^{12}) - 6 = (x^6 - 1)P(x)$$

where $P(x)$ is a polynomial in x^6. Expressing $x^6 - 1$ in terms of $g(x)$, we arrive at

$$g(x^{12}) = (x-1)g(x)P(x) + 6.$$

When this is divided by $g(x)$, the remainder is 6.

2nd Solution: Write $g(x^{12}) = g(x)Q(x) + R(x)$, where $Q(x)$ is a polynomial and $R(x)$ is the remainder we are seeking. Since the degree of the remainder is less than that of the divisor, we know that the degree of $R(x)$ is at most 4.

Since $g(x)(x-1) = x^6 - 1$, the five zeros of $g(x)$ are -1 and the other four (complex) sixth roots of unity; so if α is a zero of $g(x)$, then $\alpha^6 = 1$. Therefore

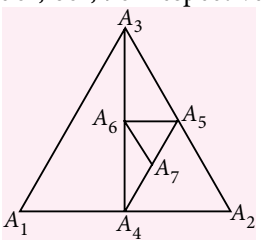
$$g(\alpha^{12}) = g[(\alpha^6)^2] = g(1) = 6$$

On the other hand,

$$g(\alpha^{12}) = g(\alpha)\, Q(\alpha) + R(\alpha), 6 = R(\alpha),$$

and this holds for five distinct values of α. But the polynomial $R(x) - 6$ of degree less than 5 can vanish at 5 places only if $R(x) - 6 = 0$ for all x, *i.e.* if $R(x) = 6$ for all x.

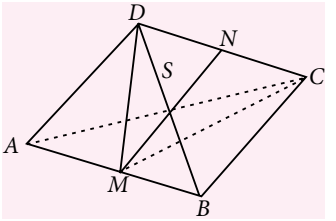
9. (e) : Triangle $A_2A_3A_4$ has vertex angles 60°, 30°, 90°, respectively. Since $\angle A_1A_2A_3 = 60°$ and A_2A_4 and A_2A_5 have the same length, $\Delta A_2A_4A_5$ is equilateral. Therefore, $\Delta A_3A_4A_5$ has vertex angles 30°, 30°, 120°, respectively. Then $\Delta A_4A_5A_6$ has vertex angles 30°, 60°, 90° respectively.



Finally, since $\angle A_4A_5A_6 = 60°$ and A_5A_6 and A_5A_7 have the same length, $\Delta A_5A_6A_7$ is again equilateral. The next cycle yields four triangles, each similar to the corresponding triangle in the previous cycle. Therefore $\Delta A_n A_{n+1} A_{n+2} \sim \Delta A_{n+4} A_{n+5} A_{n+6}$ with A_n and A_{n+4} as corresponding vertices. Thus

$$\angle A_{44}A_{45}A_{43} = \angle A_4A_5A_3 = 120°$$

10. (c) : Let M and N be the mid-points of AB and CD, respectively. We claim that M and N are the unique choices for P and Q which minimize the distance PQ. To show this we consider the set S of all points equidistant from A and B. S is the plane perpendicular to AB through M. Since C and D are equidistant from A and B, they lie in S, and so does the line through C and D. Now M is the foot of the perpendicular to AB from any point Q on CD. Therefore, if P is only point on AB, $MQ < PQ$ unless $P = M$.



Similarly, the plane through N perpendicular to CD contains AB. In particular, $MN \perp CD$; thus

$MN < MQ$ unless $Q = N$.

By transitivity, $MN < PQ$ unless $P = M$ and $Q = N$. This proves the claim.

To compute the length of MN, we note that MN is the altitude of isosceles ΔDMC with sides of lengths $\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 1$. The Pythagorean theorem now yields

$$MN = \sqrt{(MC)^2 - (NC)^2} = \sqrt{\frac{3}{4} - \frac{1}{4}}$$

$$= \frac{\sqrt{2}}{2} = \text{minimal distance } PQ.$$

11. (b) : 1st Solution: If the bug travels indefinitely, the algebraic sum of the horizontal components of its moves approaches 4/5, the limit of the geometric series

$$1 - \frac{1}{4} + \frac{1}{16} - \ldots = \frac{1}{1 - \left(-\frac{1}{4}\right)}.$$

Similarly, the algebraic sum of the vertical components of its moves approaches

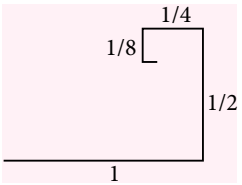
$$\frac{2}{5} = \frac{1}{2} - \frac{1}{8} + \frac{1}{32} - \ldots\ldots$$

Therefore, the bug will get arbitrarily close to $\left(\frac{4}{5}, \frac{2}{5}\right)$

2nd Solution: The line segments may be regarded as a complex geometric sequence with $a_1 = 1$ and $r = i/2$. Its sum is

$$\sum_{i=1}^{\infty} a_i = \frac{a_1}{1-r} = \frac{2}{2-i} = \frac{4+2i}{5} = \frac{4}{5} + \frac{2}{5}i$$

In coordinate language, the limit is the point $\left(\frac{4}{5}, \frac{2}{5}\right)$



Remark: The figure shows that the limiting position (x, y) of the bug satisfies $x > \frac{3}{4}, y > \frac{3}{8}$. These inequalities alone prove that (b) is the correct choice.

12. (a) : 1st Solution: For all $x \neq -3/2$,

$$x = f(f(x)) = \frac{c\left(\frac{cx}{2x+3}\right)}{2\left(\frac{cx}{2x+3}\right)+3} = \frac{c^2 x}{2cx+6x+9},$$

which implies $(2c + 6)x + (9 - c^2) = 0$. Therefore, $2c + 6 = 0$ and $9 - c^2 = 0$. Thus, $c = -3$.

2nd Solution: The condition $f[f(x)] = x$ says that the function $f(x)$ is its own inverse. Thus, if

$$y = \frac{cx}{2x+3}$$

is solved for x as a function of y, the result will be $x = f(y)$.

Indeed, we obtain

$$2xy + 3y - cx = 0, \; x(2y - c) = -3y,$$

and $x = \frac{-3y}{2y-c} = f(y)$, which implies $c = -3$.

Remark: In order to form $f[f(x)]$, we assumed that if $x \neq -3/2$, then $f(x) \neq -3/2$. This assumption can be justified expost factor now that we have found $c = -3$; indeed if

$$f(x) = \frac{-3x}{2x+3} = -\frac{3}{2},$$

then $6x = 6x + 9$, a contradiction.

13. (c) : 1st Solution: Let $f(x) = x^2 + x + 1$ and $g_n(x) = x^{2n} + 1 + (x + 1)^{2n}$. Note that $(x - 1)f(x) = x^3 - 1$, so that the zeros of $f(x)$ are the complex cube roots of 1:

$$\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i = \cos 120° + i \sin 120° = e^{\frac{i\pi}{3}},$$

$$\omega' = -\frac{1}{2} - \frac{\sqrt{3}}{2}i = \cos 240° + i \sin 240° = e^{\frac{i2\pi}{3}} = \omega^2$$

Note that $\omega^3 = (\omega')^3 = 1$.

Now $g_n(x)$ is divisible by $f(x)$ if and only if $g_n(\omega) = g_n(\omega') = 0$. Since ω and ω' are complex conjugates, it suffixes to determine those n for which

$$g_n(\omega) = \omega^{2n} + (\omega + 1)^{2n} + 1 = 0.$$

We note that

$$\omega + 1 = \frac{1}{2} + \frac{\sqrt{3}}{2}i = e^{\frac{i\pi}{6}}, \text{ so } (\omega+1)^2 = e^{\frac{i\pi}{3}} = \omega$$

Thus, whenever n is a multiple of 3, say $n = 3k$, we have

$$g_{3k}(\omega) = \omega^{6k} + (\omega + 1)^{6k} + 1$$
$$= 1 + 1 + 1 = 3 \neq 0.$$

Suppose n is not a multiple of 3.

If $n = 3k + 1$,

$$\omega^{2n} = \omega^{6k+2} = \omega^2, \ (\omega+1)^{2n} = \omega^n = \omega^{3k+1} = \omega,$$

and $g_{3k+1}(\omega) = \omega^2 + \omega + 1 = -1 + 1 = 0$.

If $n = 3k + 2$,

$$\omega^{2n} = \omega^{6k+4} = \omega, \ (\omega+1)^{2n} = \omega^{3k+2} = \omega^2$$

and $g_{3k+2}(\omega) = \omega + \omega^2 + 1 = 0$

Thus $g_n(\omega) \neq 0$ if and only if n is a multiple of 3, and $g_n(x)$ fails to be divisible by $f(x)$ only in that case.

2nd Solution: Both of the given polynomials have integer coefficients, and $x^2 + x + 1$ has leading coefficient 1. Hence if

(1) $x^{2n} + 1 + (x + 1)^{2n} = Q(x)(x^2 + x + 1)$,

then $Q(x)$ has integer coefficients. Setting $x = 2$ in (1), we obtain

$$2^{2n} + 1 + 3^{2n} = Q(2)\cdot 7,$$

which shows that 7 divides $2^{2n} + 1 + 3^{2n}$. However if n is divisible by 3, both 2^{2n} and 3^{2n} leave remainders of 1 upon division by 7, since $2^{6k} = 64^k = (7\cdot 9 + 1)^k$, and $3^{6k} = (729)^k = (7\cdot 104 + 1)^k$; so $2^{2n} + 1 + 3^{2n}$ leaves a remainder of 3 upon division by 7.

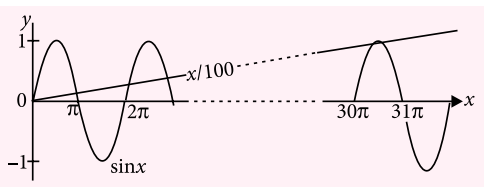
Remark: The second solution, unlike the first, does not tell us what happens when n is not divisible by 3, but it does enable us to answer the question. We know that only one listed answer is correct, and (c) is a correct answer since 3 divides 21.

14. (c) : We have, $\frac{x}{100} = \sin x$ if and only if $\frac{-x}{100} = \sin(-x)$; thus, the given equation has an equal number of positive and negative solutions. Also, $x = 0$ is a solution. Furthermore, all positive solutions are less than or equal to 100, since

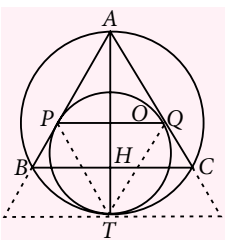
$$|x| = 100|\sin x| \leq 100$$

Since $15.5 \leq \frac{100}{(2\pi)} \leq 16$, the graphs of $\frac{x}{100}$ and $\sin x$ are as shown in the figure. Thus there is one solution to the given equation between 0 and π and two solutions in each of the intervals from $(2k - 1)\pi$ to $(2k + 1)\pi$, $1 \leq k \leq 15$. The total number of solutions is, therefore,

$$1 + 2(1 + 2\cdot 15) = 63.$$



15. (c) : Let O and H be the points at which PQ and BC, respectively, intersect diameter AT. Sides AB and AC form a portion of the equilateral triangle circumscribing the smaller circle and tangent to the smaller circle at T.



Therefore, ΔPQT is an equilateral triangle. Since ΔAPQ is an equilateral triangle with a side in common with ΔPQT, $\Delta APQ \cong \Delta PQT$. Thus $AO = OT$ and O is the centre of the larger circle. This implies

$$AO = \frac{2}{3}(AH), \text{ so that } PQ = \frac{2}{3}(BC) = 8$$

16. (b) : **1st Solution:** If $n^2 = (ab3c)_8$, let $n = (de)_8$. Then $n^2 = (8d + e)^2 = 64d^2 + 8(2de) + e^2$. Thus, the 3 in $ab3c$ is the first digit (in base 8) of the sum of the eights digit of e^2 (in base 8) and the units digit of $(2de)$ (in base 8). The latter is even, so the former is odd. The entire table of base 8 representations of squares of base 8 digits appears below.

e	1	2	3	4	5	6	7
e^2	1	4	11	20	31	44	61

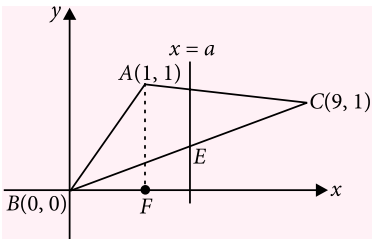
The eights digit of e^2 is odd only if e is 3 or 5; in either case c, which is the units digit of e^2, is 1. (In fact, there are three choices for n: $(33)_8$, $(73)_8$ and $(45)_8$. The squares are $(1331)_8$, $(6631)_8$ and $(2531)_8$, respectively.)

2nd Solution: We are given

$n^2 = (ab3c)_8 = 8^3a + 8^2b + 8\cdot3 + c$

If n is even, n^2 is divisible by 4, and its remainder upon division by 8 is 0 or 4. If n is odd, say $n = 2k + 1$, then $n^2 = 4(k^2 + k) + 1$, and since $k^2 + k = k(k + 1)$ is always even, n^2 has remainder 1 upon division by 8. Thus in all cases, the only possible values of c are 0, 1 or 4. If $c = 0$, then $n^2 = 8(8k + 3)$, an impossibility since 8 is not a square. If $c = 4$, then $n^2 = 4(8L + 7)$ another impossibility since no odd squares have the form $8L + 7$. Thus $c = 1$.

17. (b) : In the adjoining figure, ABC is the given triangle and $x = a$ is the dividing line. Since area $\Delta ABC = \frac{1}{2}(1)(8) = 4$, the two regions must each have area 2. Since the portion of ΔABC to the vertical line through vertex A has area less than area $\Delta ABF = \frac{1}{2}$, the line $x = a$ is indeed right of A as shown. Since the equation of line BC is $y = \frac{x}{9}$ the vertical line $x = a$ intersects BC at a point $E : \left(a, \frac{a}{9}\right)$.



Thus

Area $\Delta DEC = 2 = \frac{1}{2}\left(1 - \frac{a}{9}\right)(9 - a)$,

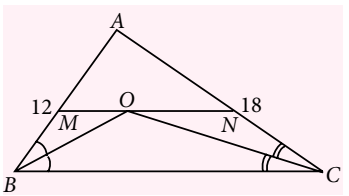
or $(9 - a)^2 = 36$

Then $9 - a = \pm 6$, and $a = 15$ or 3. Since the line $x = a$ must intersect ΔABC, $x = 3$.

18. (a) : Since MN is parallel to BC,

$\angle MOB = \angle CBO = \angle OBM$, and

$\angle CON = \angle OCB = \angle NCO$



Therefore $MB = MO$ and $ON = NC$. Hence

$AM + MO + ON + AN$

$= (AM + MB) + (AN + NC)$

$= AB + AC = 12 + 18 = 30$.

Note that the given value $BC = 24$ was not needed; in fact, only the sum of the lengths of the other two sides was used.

19. Let us show that the real roots of

$$x^2 + 2ax + \frac{1}{16} = -a + \sqrt{a^2 + x - \frac{1}{16}} \quad \ldots (1)$$

are precisely the roots of $x^2 + 2ax + \frac{1}{16} = x$ ... (2)

Note that the roots of (2) can be easily computed; they are $\frac{1}{2} - a \pm \sqrt{a^2 - a + \frac{3}{16}}$.

Since $0 < a < \frac{1}{4}$,

$$a^2 - a + \frac{3}{16} = \left(\frac{1}{4} - a\right)\left(\frac{3}{4} - a\right) > 0$$

Therefore the roots are real and distinct; it is not difficult to see that they are positive. Let α be any root of (2). Then

$$(\alpha + a)^2 = a^2 + \alpha - \frac{1}{16} \Rightarrow (\alpha + a) = \sqrt{a^2 + \alpha - \frac{1}{16}}$$

$$\Rightarrow \alpha = -a + \sqrt{a^2 + \alpha - \frac{1}{16}}$$

Therefore, $\alpha^2 + 2a\alpha + \frac{1}{16} = -a + \sqrt{a^2 + \alpha - \frac{1}{16}}$;

i.e., α is a root of (1). Conversely suppose α is a real root of (1). Let

$$\alpha^2 + 2a\alpha + \frac{1}{16} = \beta \quad \ldots (3)$$

Then $\beta = (\alpha + a)^2 - a^2 + \frac{1}{16} \geq \frac{1}{16} - a^2 > 0$.

From (3) and (1), we get

$$\beta = -a + \sqrt{a^2 + \alpha - \frac{1}{16}}$$

$$\Rightarrow (\beta + a)^2 = a^2 + \alpha - \frac{1}{16}$$

$$\Rightarrow \beta^2 + a^2 + 2a\beta = a^2 + \alpha - \frac{1}{16}$$

$$\Rightarrow \beta^2 + 2a\beta + \frac{1}{16} = \alpha \quad \ldots (4)$$

Therefore $\alpha > 0$ and

(3) – (4) $\Rightarrow$ $(\alpha - \beta)(\alpha + \beta + 2a + 1) = 0$

Since α, β, a are positive, $\alpha - \beta = 0$; *i.e.*, $\alpha = \beta$. Therefore by (4), α is a root of (2). Hence the real roots of (1) are the roots of (2), *viz.*,

$$\frac{1}{2} - a + \sqrt{a^2 - a + \frac{3}{16}} \text{ and } \frac{1}{2} - a - \sqrt{a^2 - a + \frac{3}{16}}$$

20. **1st Solution:** The given equation can be considered as a quadratic in x :

$3x^2 + (10 - 4y)x + (3y^2 - 10y + 10) = 0$.

Solving for x, we get

$$x = \frac{1}{6}\left(4y - 10 \pm \sqrt{(10-4y)^2 - 12(3y^2 - 10y + 10)}\right)$$

Since x, y are real

$(10 - 4y)^2 - 12(3y^2 - 10y + 10) \geq 0$

which on simplification gives $(y - 1)^2 \leq 0$; *i.e.*, $y = 1$ and therefore $x = -1$.

2nd Solution: L.H.S. of the given equation

$= 3(x^2 + 2x + 1) + 3(y^2 - 2y + 1) + 4(1 + x - y - xy)$

$= 3(1 + x)^2 + 3(1 - y)^2 + 4(1 + x)(1 - y)$

$= (1 + x)^2 + (1 - y)^2 + 2[(1 + x)^2 + (1 - y)^2 + 2(1 + x)(1 - y)]$

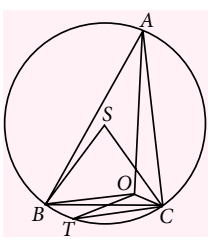
$= (1 + x)^2 + (1 - y)^2 + 2[(1 + x) + (1 - y)]^2$

Since x, y are real, $1 + x = 0 = 1 - y$;

i.e., $(x, y) = (-1, 1)$.

21. **1st Solution :** The diagonals of the quadrilateral $BTCO$ bisect each other at M. Hence $BTCO$ is a parallelogram. Since $CO \perp AB$ and $CO \parallel TB$, we have $TB \perp AB$. Similarly $TC \perp AC$. Thus the circle having AT as diameter passes through B and C. This is the circumcircle of triangle ABC. Let S be the circumcentre. Then $\angle BSC = 2\angle BAC = 60°$ and $\angle SBC = \angle SCB = 60°$; therefore ΔSBC is equilateral. Therefore BC is equal to the radius of the circle. Therefore $AT = 2BC$.

2nd Solution : Let the circumcentre of ΔABC be the origin. Let z_1, z_2, z_3 be complex numbers representing A, B, C respectively. Then the centroid G is $\frac{z_1 + z_2 + z_3}{3}$.



Since G lies on SO such that $SG : SO = 1 : 3$, O is $z_1 + z_2 + z_3$. Since M is the mid-point of BC, its coordinate is $\frac{z_2 + z_3}{2}$. Since T divides the segment OM in the ratio $2 : -1$, its coordinate is

$$\frac{1}{-1+2}\left(-1(z_1 + z_2 + z_3) + 2\left(\frac{z_2 + z_3}{2}\right)\right) = -z_1$$

$\therefore$ $AT = |-2z_1| = 2R$. Also $BC = 2R\sin 30° = R$.

22. By appolonius theorem, we have

$AB^2 + AC^2 = 2(AD^2 + BD^2)$.

This and two similar expression yield

$$\left.\begin{aligned} 2(b^2 + c^2) &= 4AD^2 + a^2, \\ 2(c^2 + a^2) &= 4BE^2 + b^2 \\ \text{and } 2(a^2 + b^2) &= 4CF^2 + c^2 \end{aligned}\right\} \quad \ldots. (1)$$

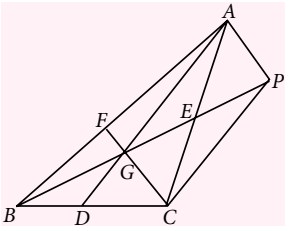
Since B, D, g, F are concyclic $AG \cdot AD = AF \cdot AB$; now using $AG = \frac{2}{3}AD$, we get

$4AD^2 = 3c^2$ ....(2)

Therefore the first relation of (1) gives

$2b^2 = c^2 + a^2$ ...(3)

Since $\angle C$ is obtuse, $c^2 > a^2 + b^2$; therefore by (3),



$$2b^2 > 2a^2 + b^2 \Rightarrow b^2 > 2a^2 \Rightarrow \frac{AC}{BC} > \sqrt{2}$$

Using (3) in the second and third relations of (1), we get

$4BE^2 = 3b^2$ and $4CF^2 = 3a^2$.

Now using these two relations and (2), we get

$$AG = \frac{2}{3}AD = \frac{c}{\sqrt{3}}, \quad GP = 2GE = \frac{2}{3}BE = \frac{b}{\sqrt{3}}$$

and $AP = GC = \frac{2}{3}CF = \frac{a}{\sqrt{3}}$

Thus $\frac{GA}{AB} = \frac{AP}{BC} = \frac{PG}{CA} = \frac{1}{\sqrt{3}}$. So the triangles GAP and ABC are similar.

23. Let $\lambda = 11 + 11\sqrt{11a^2 + 1}$

Then $(\lambda - 11)^2 = 11^2(11a^2 + 1)$;

Simplifying we get $\lambda(\lambda - 22) = 11^3a^2$.

Putting $|a| = \frac{r}{s}$ where $r, s \in N$, such that $(r, s) = 1$, gives $\lambda(\lambda - 22)s^2 = 11^3r^2$.

Since $11^2 | s$ because otherwise 11 would divide r, $11 | \lambda$. Writing $\lambda = 11\mu$, we get $\mu(\mu - 2)s^2 = 11r^2$. Since $11 | s$ for otherwise we would have $11 | r$, it follows that $s = 1$. Thus we have $\mu(\mu - 2) = 11r^2$. Since $\mu - 2$ and μ are consecutive odd integers, they are relatively prime. If $11 | \mu - 2$, then μ is a square of the form $11n + 2$ which is not possible. Therefore $11|\mu$ and hence $\mu = 11n^2$ for some $n \in N$. Thus we have $\lambda = 11\mu = 11^2n^2$.

24. We are to find an n such that 10^{1993} is the highest power of 10 dividing $n!$. Since multiples of 2 occur far more often than multiples of 5 it is enough to find an n such that 5^{1993} is the maximum power of 5 that divides $n!$, that is we have to solve for n in the equation:

$$1993 = \left[\frac{n}{5}\right] + \left[\frac{n}{25}\right] + \left[\frac{n}{125}\right] + \ldots$$

Now,

$$\left[\frac{n}{5}\right] + \left[\frac{n}{25}\right] + \left[\frac{n}{125}\right] + \ldots \le \frac{n}{5} + \frac{n}{25} + \frac{n}{125} + \ldots$$

$$\le \frac{n}{5}\left[1 + \frac{1}{5} + \frac{1}{25} + \ldots\right] \le \frac{n}{5}\left(\frac{5}{4}\right) = \frac{n}{4}$$

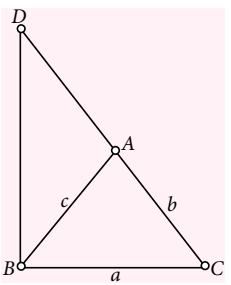
i.e., $n \ge 7972$. Since

$$\left[\frac{7972}{5}\right] + \left[\frac{7972}{25}\right] + \left[\frac{7972}{125}\right] + \left[\frac{7972}{625}\right] + \left[\frac{7972}{3125}\right] = 1989$$

We have to arrange for four more multiples of 5. Note that 7975! will have one more multiple of 5 and one more multiple of 25 than 7972! and 7985! will have 3 more multiples of 5 and 1 more multiple of 25 (that is, in all 4 more multiples of 5) than 7972!. Thus 7985 is the required number. Note that 7986, 7987, 7988 and 7989 also satisfy the requirement.

25. **1st Solution:**

First assume that in the triangle ABC, $A = 2B$. Produce CA to D such that $AD = AB$. Join BD.



By construction, it is clear that ABC is an isosceles triangle and so

$$\angle ADB = \angle ABD$$

But $\angle ADB + \angle ABD = \angle BAC$ (the external angle)

Hence, $\angle ADB = \angle ABD = \frac{A}{2} = B$

In triangles ABC and BDC we have $\angle ABC = \angle BDC$ and $\angle C$ is common. So ΔABC is similar to ΔBDC. Therefore,

$$\frac{AC}{BC} = \frac{BC}{DC}.$$

It follows that $a^2 = b(b + c)$

Now we prove the converse. Assume that $a^2 = b(b + c)$. We refer to the same figure. As before, in the isosceles triangle ABD, we have

$$\angle ABD = \angle ADB.$$

So each of these angles is equal to half of their sum which is A. Thus, in particular,

$$\angle ADB = \frac{A}{2} \qquad \ldots. (1)$$

On the other hand, in triangles ACB and BCD, we have, as a consequence of the assumption $a^2 = b(b + c)$,

$$\frac{AC}{BC} = \frac{BC}{DC}, \text{ and } \angle C \text{ is common.}$$

So the two triangles are similar and the result follows:

$$\angle CDB = \angle CBA = \angle B \qquad \ldots.(2)$$

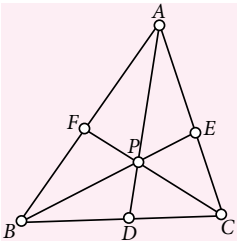
From (1) and (2), it follows that $B = A/2$, as desired.

2nd Solution: We may use the sine rule for a triangle to dispose of both the implications simultaneously.

$$\begin{aligned} A = 2B &\Leftrightarrow A - B = B \\ &\Leftrightarrow \sin(A - B) = \sin B \\ &\Leftrightarrow \sin(A - B)\sin(A + B) = \sin B \sin C \\ &\Leftrightarrow \sin^2 A - \sin^2 B = \sin B \sin C \\ &\Leftrightarrow (2R\sin A)^2 - (2R\sin B)^2 \\ &\qquad = (2R\sin B)(2R\sin C) \\ &\Leftrightarrow a^2 - b^2 = bc \\ &\Leftrightarrow a^2 = b(b + c) \end{aligned}$$

26. **1st Solution:** We use the fact that the areas of two triangles having the same height are in the ratio of their bases. We also use some simple properties of equal fractions.

Specifically, if $\frac{a}{b} = \frac{c}{d}$, then each fraction is also equal to $\frac{(a+c)}{(b+d)}$ as well as $\frac{(a-c)}{(b-d)}$.



Now, $\dfrac{[ACF]}{[BCF]} = \dfrac{AF}{FB} = \dfrac{[APF]}{[BPF]}$.

So, $\dfrac{AF}{FB} = \dfrac{[ACF]-[APF]}{[BCF]-[BPF]} = \dfrac{[ACP]}{[BCP]}$ .... (1)

Similarly from

$$\frac{[ABE]}{[CBE]} = \frac{AE}{EC} = \frac{[APE]}{[CPE]}$$

one obtains,

$$\frac{AE}{EC} = \frac{[ABE]-[APE]}{[CBE]-[CPE]} = \frac{[ABP]}{[CBP]} \quad \text{....(2)}$$

From (1) and (2), by addition, we get

$$\frac{AF}{FB} + \frac{AE}{EC} = \frac{[ACP]+[ABP]}{[BCP]} \quad \text{.....(3)}$$

Again,

$$\frac{AP}{PD} = \frac{[ABP]}{[DBP]} = \frac{[CAP]}{[DCP]} = \frac{[ABP]+[ACP]}{[BCP]} \quad \text{....(4)}$$

From (3) and (4), we have the desired result.

2nd Solution: Applying Ceva's theorem to the Cevians AD, BE, CF which are concurrent at P, we have

$$\frac{AF}{FB}\cdot\frac{BC}{DC}\cdot\frac{CE}{EA} = 1$$

Therefore, $\dfrac{AE}{EC} = \dfrac{AF}{FB}\cdot\dfrac{BD}{DC}$

Hence,

$$\frac{AF}{FB} + \frac{AE}{AC} = \frac{AF}{FB}\left(1 + \frac{BD}{DC}\right) = \frac{AF}{FB}\cdot\frac{BC}{DC} \quad \text{.....(1)}$$

Now applying 'Menelaus' Theorem to triangle ABD, whose sides are cut by the line FPC, we have

$$\frac{AF}{FB}\cdot\frac{BC}{DC}\cdot\frac{DP}{PA} = 1$$

Consequently, $\dfrac{AF}{FB}\cdot\dfrac{BC}{DC} = \dfrac{AP}{PD}$ .....(2)

Comparing (1) and (2), we have the desired relation.

27. If both the urns have the same number of balls, then we can empty both the urns in one operation. Else, we remove the same number of balls from each of the urns so that one of the urns contains exactly one ball. (If m and n denote the number of balls in the urns, and say $m > n$, then take out $n - 1$ balls from each.) We now double the number of balls of the urn which contains only one ball and remove one ball from each of the urn. This process decreases the number of balls in the other urn by 1. Continuing this way we reach a stage when both the urns contain one ball each whence we can empty the urns removing one ball from each of the two urns.

28. Since $x + 1$ divides $ax^2 + bx + c$, we must have $a + c = b$. Thus we have to count the number of triples (a, b, c) with the condition that a, b, c lie in the set $\{1, 2, 3, ..., 1999\}$, $a \neq c$ and $a + c = b$. If we take $a < c$, then for each a with $1 \le a \le 999$, c can take values from $a + 1$ to $1999 - a$. Thus for $a = 1$, c runs from 2 to 1998 giving 1997 ordered pairs (a, c) with $a < c$; for $a = 2$, c runs from 3 to 1997, giving 1995 pairs (a, c) with $a < c$, and so on. The number of ordered pairs (a, c) with $a < c$ and $a + c$ lying in the set $\{1, 2, 3, ..., 1999\}$ is thus equal to

$$1997 + 1995 + 1993 + ... + 1 = 999^2.$$

Similarly the number of pairs (a, c) with $c < a$ and $a + c$ lying in the set $\{1, 2, 3, ..., 1999\}$ is 999^2. Hence the required number of polynomials is $2 \cdot 999^2 = 1996002$.

29. Let $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ and x_8 be the numbers written at the corners. Then, the final sum is given by

$$\sum_{i=1}^{8} x_i + x_1x_2x_3x_4 + x_5x_6x_7x_8 + x_1x_4x_5x_8 + x_2x_3x_6x_7 + x_1x_2x_5x_6 + x_3x_4x_7x_8$$

Because there are fourteen terms in the above sum and each of the terms is + 1 or – 1, the sum will be zero only if some seven terms are + 1 each and the remaining seven terms are – 1 each.

But, the product of the fourteen terms is

$$(x_1x_2x_3x_4x_5x_6x_7x_8)^4 = (\pm 1)^4 = +1.$$

Therefore, it is impossible to have an odd number of –1's in the above sum.

We conclude that the desired arrangement is not possible. ■■

SAMPLE PAPER

CBSE Board 2015

Time : 3 hrs. Marks : 100

GENERAL INSTRUCTIONS

(i) All questions are compulsory.

(ii) Th e question paper consists of 26 questions divided into three sections A, B and C. Section A comprises of 6 questions of one mark each, section B comprises of 13 questions of four marks each and section C comprises of 7 questions of six marks each.

(iii) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.

(iv) Th ere is no overall choice. However, internal choice has been provided in 5 questions of four marks each and 2 questions of six marks each. You have to attempt only one of the alternatives in all such questions.

(v) Use of calculators is not permitted. You may ask for logarithmic tables, if required.

SECTION-A

1. If A is a singular matrix, find x.

$$A=\begin{bmatrix} x-1 & 0 & 2 \\ -1 & -2 & x-3 \\ 2 & 0 & 4 \end{bmatrix}$$

2. What is the value of $\tan^{-1}2 + \tan^{-1}3$.

3. Let $\vec{a}=5\hat{i}-\hat{j}+8\hat{k}$ and $\vec{b}=\hat{i}+\hat{j}+\lambda\hat{k}$. If the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are orthogonal to each other, then find the value of $|\lambda|$.

4. Solve the differential equation $\sin\left(\frac{dy}{dx}\right)=a$.

5. Write the value of $(\hat{k}\times\hat{j})\cdot\hat{i}+\hat{j}\cdot\hat{k}$.

6. Find the distance of the plane $2x - y + 2z + 1 = 0$ from the origin.

SECTION-B

7. Show that $f: N \to N$, defined by

$$f(n)=\begin{cases}\frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even}\end{cases}$$

is a many-one onto function.

8. Prove that $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix}=a^3$

OR

Show that $f(x) = x^{4/3}$ is differentiable at $x = 0$, and hence find $f'(0)$.

9. Prove that $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)=\left(\frac{\pi}{4}-\frac{x}{2}\right)$.

10. Given $x = cy + bz$, $y = az + cx$, $z = bx + ay$ where, x, y, z are not all zero. Prove that:
$a^2 + b^2 + c^2 + 2abc = 1$.

OR

Find two positive numbers whose sum is 24 and whose product is maximum.

11. Evaluate: $\int\frac{\cos x}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^3}dx$

OR

Evaluate: $\int_2^4\frac{(x^2+x)}{\sqrt{2x+1}}dx$

12. Two schools A and B decide to award prizes to their students for three values honesty (x),

punctuality (y) and obedience (z). School A desires to award a total of ₹ 11000 for the three values to 5, 4 and 3 students respectively while school B wishes to award ₹ 10700 for the three values to 4, 3 and 5 students respectively. If all the three prizes together amount to ₹ 2700, then

(i) represent the above situation by a matrix equation and form linear equations using matrix multiplication.

(ii) is it possible to solve the system of equations so obtained using matrices?

(iii) which value you prefer to be rewarded most and why?

13. Find the area bounded by the line $y = x$, the x-axis and the abscissa $x = -1$, $x = 2$.

OR

Solve: $(x\sqrt{x^2+y^2} - y^2)dx + xy\,dy = 0$

14. X is taking up subjects : Mathematics, Physics and Chemistry in the examination. The probabilities of getting grade A in these subjects are 0.2, 0.3 and 0.5, respectively. Find the probability that he gets

(i) grade A in no subject.

(ii) grade A in two subjects.

15. Find the equation of the plane passing through the intersection of the planes $\vec{r}\cdot(2\hat{i}+\hat{j}+3\hat{k})=7$, $\vec{r}\cdot(2\hat{i}+5\hat{j}+3\hat{k})=9$ and the point (2, 1, 3).

16. Find a point on the curve $y = (x - 2)^2$, at which the tangent is parallel to the chord joining the points (2, 0) and (4, 4).

17. If $y = x^{x^x}$, then find $\frac{dy}{dx}$.

18. Find the adjoint of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and verify that $A[\text{adj}\,(A)] = |A|\,I$.

19. Prove that $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right), x \geq 0$.

OR

Solve the equation :

$$\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$$

SECTION-C

20. Prove that

$$2\tan^{-1}\left[\sqrt{\frac{a-b}{a+b}}\tan\frac{\theta}{2}\right] = \cos^{-1}\left[\frac{b+a\cos\theta}{a+b\cos\theta}\right]$$

21. The male-female ratio of a village increases continuously at the rate proportional to the ratio at any time. If the ratio of male-female of the villages was 1000 : 980 in 1999 and 1000 : 950 in 2009, then what will be the ratio in 2019?

(i) Why gender equality is value for society?

(ii) What should society do to reduce the male-female ratio to 1?

22. A factory owner wants to purchase two types of machines, A and B, for his factory. The machine A requires an area of 1000 m^2 and 12 skilled men for running it and its daily output is 50 units, whereas the machine B required 1200 m^2 area and 8 skilled men, and its daily output is 40 units. If an area of 7600 m^2 and 72 skilled men be available to operate the machine, how many machines A and B respectively should be purchased to maximize the daily output?

OR

Find the shortest distance between the lines $\vec{r} = (8+3\lambda)\hat{i} - (9+16\lambda)\hat{j} + (10+7\lambda)\hat{k}$ and $\vec{r} = 15\hat{i} + 29\hat{j} + 5\hat{k} + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$.

23. Coloured balls are distributed in three bags as shown in the following table :

Bag	Colour of ball		
	Green	Black	Blue
I	1	2	3
II	2	4	1
III	3	3	2

A bag is selected at random and then two balls are drawn from the selected bag. Find the probability that the balls drawn are black and blue.

OR

The parabola $y^2 = 2x$ divides the circle $x^2 + y^2 = 8$ in two parts. Then, what is the ratio of the areas of these parts?

24. Find the volume of the largest cylinder that can be inscribed in a sphere of radius r cm.

25. Find x, if

$$\tan^{-1}\left(\frac{x+1}{x-1}\right)+\tan^{-1}\left(\frac{x-1}{x}\right)=\tan^{-1}(-7).$$

26. Evaluate : $\int \frac{3\cos x+2}{\sin x+2\cos x+3}\,dx$

SOLUTIONS

1. A is singular $\therefore$ $|A| = 0$

$|A| = (x-1)(-8-0)+2(4) = 0$

$\Rightarrow \quad -8x+8+8=0 \Rightarrow x=2$

2. Here, $2\times 3 = 6 > 1$

$$\therefore \quad \tan^{-1}2+\tan^{-1}3 = \pi+\tan^{-1}\left(\frac{2+3}{1-2\cdot 3}\right)$$

$$= \pi+\tan^{-1}(-1)=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$$

3. We have, $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=0$

$\Rightarrow |\vec{a}|^2-|\vec{b}|^2=0 \Rightarrow 90=2+\lambda^2 \Rightarrow |\lambda|=\sqrt{88}$

4. Given, $\sin\left(\frac{dy}{dx}\right)=a \Rightarrow \frac{dy}{dx}=\sin^{-1}(a)$

$\Rightarrow \quad dy = \sin^{-1}(a)\,dx$

On integrating both sides, we get

$$y = x\sin^{-1}(a)+c$$

5. We have,

$(\hat{k}\times\hat{j})\cdot\hat{i}+\hat{j}\cdot\hat{k}=(-\hat{i})\cdot\hat{i}+\hat{j}\cdot\hat{k} = -(1)+0=-1$

6. Given plane is $2x - y + 2z + 1 = 0$ ...(i)

Distance of (i) from origin

$$=\left|\frac{(2\times 0)-(1\times 0)+(2\times 0)+1}{\sqrt{(2)^2+(-1)^2+(2)^2}}\right|=\frac{1}{3}$$

7. We have ,

$f(1)=\frac{(1+1)}{2}=\frac{2}{2}=1$ and $f(2)=\frac{2}{2}=1$

Thus, $f(1) = f(2)$ while $1 \neq 2$.

$\therefore$ f is many-one.

In order to show that f is onto, consider an arbitrary element $n \in N$.

If n is odd, then $(2n-1)$ is odd and

$$f(2n-1)=\frac{(2n-1+1)}{2}=\frac{2n}{2}=n$$

If n is even, then $2n$ is even and

$$f(2n)=\frac{2n}{2}=n$$

Thus, for each $n \in N$ (whether even or odd) there exists its pre-image in N.

$\therefore$ f is onto.

Hence, f is many-one onto.

8. Let $\Delta=\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix}$

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, we get

$$\Delta=\begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 3a & 7a+3b \end{vmatrix}$$

Applying $R_3 \to R_3 - 3R_2$, we get

$$\Delta=\begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 0 & a \end{vmatrix}$$

Expanding along C_1, we get

$\Delta = a \times (a^2 - 0) = a^3$ Hence proved.

OR

We have,

$$Rf'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

$$\lim_{h\to 0}\frac{h^{4/3}-0}{h}=\lim_{h\to 0}\frac{h^{4/3}}{h}=\lim_{h\to 0}h^{1/3}=0$$

and

$$Lf'(0)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{f(-h)-0}{-h}$$

$$=\lim_{h\to 0}\frac{(-h)^{4/3}-0}{(-h)}=\lim_{h\to 0}\frac{(-h)^{4/3}}{(-h)}=\lim_{h\to 0}(-h)^{1/3}=0$$

Thus, $Rf'(0) = Lf'(0) = 0$

Hence $f(x) = x^{4/3}$ is differentiable at $x = 0$ and $f'0) = 0$.

9. We have, L.H.S. $=\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$

$$=\tan^{-1}\left\{\frac{\sin\left(\frac{\pi}{2}-x\right)}{1+\cos\left(\frac{\pi}{2}-x\right)}\right\}$$

$$= \tan^{-1}\left\{\frac{2\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right\}$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\} = \left(\frac{\pi}{4}-\frac{x}{2}\right) = \text{R.H.S.}$$

$$\therefore \quad \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right) = \left(\frac{\pi}{4}-\frac{x}{2}\right)$$

10. We are given that there is a non-zero solution of the equation $x = cy + bz$, $y = az + cx$, $z = bx + ay$. This means that the system of homogeneous equations

$x - cy - bz = 0$

$cx - y + az = 0$

$bx + ay - z = 0$

has non-trivial solution.

$$\Rightarrow \begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0 \Rightarrow 1(1-a^2) - (-c)(-c-ab) + (-b)(ca+b) = 0$$

$\Rightarrow 1 - a^2 - c^2 - abc - abc - b^2 = 0$

$\Rightarrow a^2 + b^2 + c^2 + 2abc = 1$ Hence proved.

OR

Let the two positive numbers be x and $24 - x$ and let $y = x(24 - x)$, $0 < x < 24$

$\Rightarrow \quad y = 24x - x^2$ ...(i)

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = 24 - 2x \qquad \text{...(ii)}$$

For maxima/minima,

$$\frac{dy}{dx} = 0 \Rightarrow 24 - 2x = 0 \Rightarrow x = 12$$

Differentiating (ii) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -2 < 0 \text{ at } x = 12$$

$\Rightarrow$ y has a local maximum value at $x = 12$.

So, the required parts are 12 and $24 - 12 = 12$, *i.e.*, 12 and 12.

11. We have

$$\int \frac{\cos x}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^3}dx = \int \frac{\cos^2(x/2)-\sin^2(x/2)}{\{\cos(x/2)+\sin(x/2)\}^3}dx$$

$$= \int \frac{\cos(x/2)-\sin(x/2)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}dx = 2\int \frac{1}{t^2}dt,$$

where $t = \cos\frac{x}{2} + \sin\frac{x}{2}$

$$= \frac{-2}{t} + C = \frac{-2}{\cos(x/2)+\sin(x/2)} + C$$

OR

Integrating by parts, we get

$$\int_2^4 \frac{(x^2+x)}{\sqrt{2x+1}}dx = \left[(x^2+x)\cdot\sqrt{2x+1}\right]_2^4 - \int_2^4 (2x+1)\cdot\sqrt{2x+1}\,dx$$

$$= (60-6\sqrt{5}) - \int_2^4 (2x+1)^{3/2}dx$$

$$= (60-6\sqrt{5}) - \frac{1}{5}\cdot\left[(2x+1)^{5/2}\right]_2^4$$

$$= (60-6\sqrt{5}) - \left(\frac{243}{5} - 5\sqrt{5}\right)$$

$$= \left(\frac{57}{5} - \sqrt{5}\right) = \left(\frac{57-5\sqrt{5}}{5}\right)$$

12. (i) We can represent the given problem in matrix form as follows:

$$\begin{bmatrix} 5 & 4 & 3 \\ 4 & 3 & 5 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11000 \\ 10700 \\ 2700 \end{bmatrix}$$

$\therefore$ $5x + 4y + 3z = 11000$, $4x + 3y + 5z = 10700$, $x + y + z = 2700$

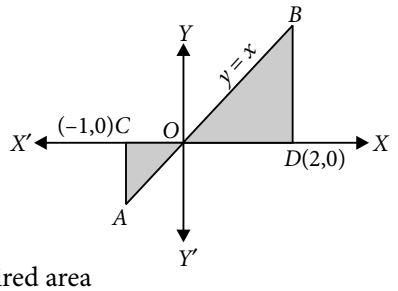
(ii) Since $\begin{vmatrix} 5 & 4 & 3 \\ 4 & 3 & 5 \\ 1 & 1 & 1 \end{vmatrix} = -3 \neq 0$

therefore, it is possible to solve the system of equations obtained in (i) by using matrices.

(iii) Preference should be given to honesty as an honest person is expected to follow all ethical values such as punctuality, justice, equality and obedience etc.

13. We know that $y = x$ is the line passing through the origin.

Now, we have to find the area of the shaded region.



Required area

= (area ΔDBO) + (area ΔACO)

$$= \int_0^2 y\,dx + \int_{-1}^{0} (-y)dx \quad [\because \text{ area } OACO \text{ is below the } x\text{-axis}]$$

$$= \int_0^2 x\,dx + \int_{-1}^{0} (-x)dx$$

$$= \left[\frac{x^2}{2}\right]_0^2 + \left[\frac{-x^2}{2}\right]_{-1}^{0} = \left[2 + \frac{1}{2}\right] = \frac{5}{2} \text{ sq. units}$$

OR

The given equation may be written as

$$\frac{dy}{dx} = \frac{y^2 - x\sqrt{x^2 + y^2}}{xy}, \text{ which is clearly homogeneous.}$$

Putting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$, we get

$$v + x\frac{dv}{dx} = \frac{v^2x^2 - x\sqrt{x^2 + v^2x^2}}{vx^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \left(\frac{v^2 - \sqrt{1+v^2}}{v} - v\right)$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{-\sqrt{1+v^2}}{v} \Rightarrow \int \frac{v}{\sqrt{1+v^2}}dv = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \sqrt{1+v^2} = -\log|x| + C$$

$$\Rightarrow \quad \sqrt{x^2 + y^2} + x\log|x| = Cx$$

14. Let the event of getting grade A in Mathematics, Physics and Chemistry be represented by M, P and C, respectively.

$P(M) = 0.2$, $P(P) = 0.3$ and $P(C) = 0.5$

$P(\bar{M}) = 0.8$, $P(\bar{P}) = 0.7$ and $P(\bar{C}) = 0.5$

(i) P(grade A in no subject)

$= P(\bar{M}) \cdot P(\bar{P}) \cdot P(\bar{C})$

$= 0.8 \times 0.7 \times 0.5 = 0.28$

(ii) P(grade A in two subjects)

$= P(M) \cdot P(P) \cdot P(\bar{C}) + P(M) \cdot P(\bar{P}) \cdot P(C) + P(\bar{M}) \cdot P(P) \cdot P(C)$

[$\because$ $M, P, C, \bar{M}, \bar{P}$ and $\bar{C}$ are independent events]

$= (0.2 \times 0.3 \times 0.5) + (0.2 \times 0.7 \times 0.5) + (0.8 \times 0.3 \times 0.5)$

$= 0.03 + 0.07 + 0.12 = 0.22$

15. Given equation of planes in cartesian form are

$2x + y + 3z - 7 = 0$ ...(i)

and $2x + 5y + 3z - 9 = 0$ ...(ii)

Equation of plane passing through the intersection of planes (i) and (ii) is

$(2x + y + 3z - 7) + \lambda(2x + 5y + 3z - 9) = 0$...(iii)

Plane (iii) passes through the point (2, 1, 3)

$\therefore \quad (4 + 1 + 9 - 7) + \lambda(4 + 5 + 9 - 9) = 0$

$\Rightarrow \quad 7 + 9\lambda = 0 \Rightarrow \lambda = \frac{-7}{9}$

On putting the value of λ in (iii), we get

$$(2x + y + 3z - 7) + \left(\frac{-7}{9}\right)(2x + 5y + 3z - 9) = 0$$

$$\Rightarrow \frac{18x + 9y + 27z - 63 - 14x - 35y - 21z + 63}{9} = 0$$

$\Rightarrow 4x - 26y + 6z = 0 \Rightarrow 2x - 13y + 3z = 0$

Equation of plane in vector form is

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} - 13\hat{j} + 3\hat{k}) = 0$$

$\therefore \quad \vec{r} \cdot (2\hat{i} - 13\hat{j} + 3\hat{k}) = 0$

16. Given curve is $y = (x - 2)^2$ ...(i)

On differentiating (i) both sides w.r.t. x, we get

$$\frac{dy}{dx} = 2(x - 2)$$

Slope of the chord joining the points (2, 0) and (4, 4) $= \frac{4-0}{4-2} = \frac{4}{2} = 2$ ...(ii)

Since, the tangent is parallel to the chord joining the points.

$\therefore \quad \frac{dy}{dx}$ = Slope of the chord

$\Rightarrow \quad 2(x - 2) = 2 \Rightarrow x - 2 = 1 \Rightarrow x = 3$

On putting $x = 3$ in (i), we get $y = 1$

Hence, the required point is (3, 1).

17. Given, $y = x^{xx}$

Taking log on both sides, we get

$\log y = x^x \log x$

Again, taking log on both sides, we get

$\log(\log y) = \log(x^x \log x)$

$\Rightarrow \log(\log y) = \log x^x + \log(\log x)$

$\Rightarrow \log(\log y) = x(\log x) + \log(\log x)$ ...(i)

On differentiating (i) both sides w.r.t. x, we get

$$\Rightarrow \frac{1}{\log y}\cdot\frac{1}{y}\cdot\frac{dy}{dx} = \left[x\cdot\frac{1}{x} + \log x\cdot 1\right] + \left[\frac{1}{\log x}\cdot\frac{1}{x}\right]$$

$$\Rightarrow \frac{1}{y\log y}\cdot\frac{dy}{dx} = 1 + \log x + \frac{1}{x\log x}$$

$$\Rightarrow \frac{dy}{dx} = y\log y\left[1 + \log x + \frac{1}{x\log x}\right]$$

$$= x^{(x^x)}\log x^{(x^x)}\left[1 + \log x + \frac{1}{x\log x}\right]$$

18. We have, $A = \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$

$\text{adj}(A) = \begin{bmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{bmatrix}'$ where C_{ij}'s are the cofactors of a_{ij}'s

$C_{11} = (-1)^2(4) = 4$

$C_{12} = (-1)^3(3) = -3$

$C_{21} = (-1)^3(2) = -2$

$C_{22} = (-1)^4(1) = 1$

$$\therefore \text{adj}(A) = \begin{bmatrix}4 & -2\\ -3 & 1\end{bmatrix} \text{ and}$$

$$|A| = \begin{vmatrix}1 & 2\\ 3 & 4\end{vmatrix} = 4 - 6 = -2 \quad \text{...(i)}$$

Now,

$$A[\text{adj}(A)] = \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}4 & -2\\ -3 & 1\end{bmatrix} = \begin{bmatrix}4-6 & -2+2\\ 12-12 & -6+4\end{bmatrix}$$

$$= \begin{bmatrix}-2 & 0\\ 0 & -2\end{bmatrix} = -2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = |A|I \quad [\text{from (i)}]$$

19. R.H.S. $= \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$

Put $x = \tan^2\theta \Rightarrow \theta = \tan^{-1}\sqrt{x}$

$$\therefore \text{R.H.S.} = \frac{1}{2}\cos^{-1}\left(\frac{1-\cos^2\theta}{1+\cos^2\theta}\right)$$

$$= \frac{1}{2}\cos^{-1}(\cos 2\theta) = \frac{1}{2}\times 2\theta = \theta = \tan^{-1}\sqrt{x}$$

= L.H.S.

OR

Given equation is

$$\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left\{\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\frac{x-1}{x-2}+\frac{x+1}{x+2}}\right\} = \frac{\pi}{4}$$

$$\Rightarrow \frac{(x-1)(x+2)+(x+1)(x-2)}{(x-2)(x+2)-(x-1)(x+1)} = \tan\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \frac{x^2-x+2x-2+x^2+x-2x-2}{(x^2-4)-(x^2-1)} = 1$$

$\Rightarrow 2x^4 - 4 = -3 \Rightarrow 2x^2 = 1$

$$\Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}}$$

20. Let $x = \tan^{-1}\left[\sqrt{\frac{a-b}{a+b}}\tan\frac{\theta}{2}\right]$

$$\Rightarrow \tan x = \sqrt{\frac{a-b}{a+b}}\tan\frac{\theta}{2}$$

On squaring both sides, we get

$$\tan^2 x = \frac{a-b}{a+b}\tan^2\frac{\theta}{2}$$

$$\text{Now, } \cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x} = \frac{1-\left(\frac{a-b}{a+b}\right)\tan^2\frac{\theta}{2}}{1+\left(\frac{a-b}{a+b}\right)\tan^2\frac{\theta}{2}}$$

$$= \frac{\left[a+b-a\tan^2\frac{\theta}{2}+b\tan^2\frac{\theta}{2}\right]}{\left[a+b+a\tan^2\frac{\theta}{2}-b\tan^2\frac{\theta}{2}\right]}$$

$$= \frac{a\left(1-\tan^2\frac{\theta}{2}\right)+b\left(1+\tan^2\frac{\theta}{2}\right)}{a\left(1+\tan^2\frac{\theta}{2}\right)+b\left(1-\tan^2\frac{\theta}{2}\right)}$$

$$= \frac{a\left[\dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right]+b}{a+b\left[\dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right]}$$

$$\Rightarrow \cos 2x = \frac{a\cos\theta + b}{a + b\cos\theta}$$

$$\Rightarrow 2x = \cos^{-1}\left[\frac{b + a\cos\theta}{a + b\cos\theta}\right]$$

$$\therefore\ 2\tan^{-1}\left[\sqrt{\frac{a-b}{a+b}}\tan\frac{\theta}{2}\right] = \cos^{-1}\left[\frac{b + a\cos\theta}{a + b\cos\theta}\right]$$

21. Let male-female ratio at any time be r.

Given $\frac{dr}{dt} \propto r \Rightarrow \frac{dr}{dt} = kr$

We have $\frac{dr}{r} = kdt$

Integrating both sides, we get

$\log r = kt + \log c$

(where $\log c$ is the constant of integration)

$\Rightarrow \log r - \log c = kt \Rightarrow \log\frac{r}{c} = kt$

$\therefore\ r = ce^{kt}$ ...(i)

$\therefore$ Let us start reckoning time from the year 1999 for the problem.

So in 1999, $t = 0$ and $r = \frac{1000}{980} = \frac{50}{49}$

Substituting in (i), we get

$$\frac{50}{49} = c \cdot e^0 \Rightarrow c = \frac{50}{49}$$

$\therefore$ (i) becomes, $r = \frac{50}{49}e^{kt}$ ...(ii)

Also in the year 2009, $t = 10$ and $r = \frac{1000}{950} = \frac{20}{19}$

Substituting in (ii), we get

$$\frac{20}{19} = \frac{50}{49}e^{10k} \Rightarrow e^{10k} = \frac{98}{95}$$

Substituting in (ii), we get

$$r = \frac{50}{49}\cdot(e^{10k})^{\frac{t}{10}} = \frac{50}{49}\left(\frac{98}{95}\right)^{\frac{t}{10}} \quad ...(iii)$$

In the year 2019, $t = 20$

$$\therefore\ r = \frac{50}{49}\left(\frac{98}{95}\right)^{\frac{20}{10}} = \frac{50}{49}\times\frac{98}{95}\times\frac{98}{95}$$

$$= \frac{100\times 98}{95\times 95} \approx 1.085 \approx 1085:1000.$$

Thus in the year 2019, the male-female ratio will be 1085 : 1000.

(i) Gender equity promotes economic growth, fertility, child mortality and under nutrition.

(ii) (a) Stop female-foeticide.
(b) Empower women to realise their rights.
(c) Provide special opportunities to women to come at par with men in all spheres of life.

22. Let the number of machine A be x and number of machine B be y. Let z be the daily output.

Now given information can be summarized as:

	A (x)	**B (y)**	**Maximum available capacity**
Area (m^2)	1000	1200	7600
Man power	12	8	72
Output	50	40	

According to question, x and y must satisfy the following conditions :

(Area) $1000x + 1200y \le 7600 \Rightarrow 5x + 6y \le 38$

(Man power) $12x + 8y \le 72 \Rightarrow 3x + 2y \le 18$

$x \ge 0, y \ge 0$

Mathematical formulation of the LPP is

Maximize $z = 50x + 40y$

subject to constraints :

$5x + 6y \le 38$

$3x + 2y \le 18$

$x \ge 0, y \ge 0$

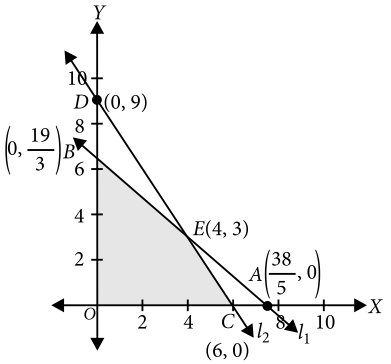
Now, we draw the lines

$l_1 : 5x + 6y = 38$

$l_2 : 3x + 2y = 18$

$l_3 : x = 0$ and $l_4 : y = 0$

Lines l_1 and l_2 meet at $E(4, 3)$.



The shaded region $OCEB$ is the feasible region which is bounded.

Vertices of the feasible region are

$$O(0, 0),\ C(6, 0),\ E(4, 3) \text{ and } B\left(0, \frac{19}{3}\right)$$

Maximize $z = 50x + 40y$

At O, $z = 50 \times 0 + 40 \times 0 = 0$

At C, $z = 50 \times 6 + 40 \times 0 = 300$

At E, $z = 50 \times 4 + 40 \times 3 = 320$

At B, $z = 50 \times 0 + 40 \times \dfrac{19}{3} = 253.33$

Clearly, the maximum output = 320 is at $E(4, 3)$, *i.e.*, when 4 machines A and 3 machines B are purchased.

OR

Given, equations of lines

$$\vec{r} = (8 + 3\lambda)\hat{i} - (9 + 16\lambda)\hat{j} + (10 + 7\lambda)\hat{k}$$

i.e., $\vec{r} = 8\hat{i} - 9\hat{j} + 10\hat{k} + \lambda(3\hat{i} - 16\hat{j} + 7\hat{k})$

and $\vec{r} = 15\hat{i} + 29\hat{j} + 5\hat{k} + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$

Here, $\vec{a}_1 = 8\hat{i} - 9\hat{j} + 10\hat{k},\ \vec{b}_1 = 3\hat{i} - 16\hat{j} + 7\hat{k}$

$\vec{a}_2 = 15\hat{i} + 29\hat{j} + 5\hat{k},\ \vec{b}_2 = 3\hat{i} + 8\hat{j} - 5\hat{k}$

$$\vec{a}_2 - \vec{a}_1 = 7\hat{i} + 38\hat{j} - 5\hat{k}$$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix}$$

$$= \hat{i}(80 - 56) - \hat{j}(-15 - 21) + \hat{k}(24 + 48)$$

$$= (24\hat{i} + 36\hat{j} + 72\hat{k})$$

$$\therefore \quad \text{Shortest distance, } d = \left|\frac{(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}\right|$$

$$= \left|\frac{(7\hat{i} + 38\hat{j} - 5\hat{k}) \cdot (24\hat{i} + 36\hat{j} + 72\hat{k})}{\sqrt{(24)^2 + (36)^2 + (72)^2}}\right|$$

$$= \left|\frac{168 + 1368 - 360}{\sqrt{576 + 1296 + 5184}}\right| = \left|\frac{1176}{\sqrt{7056}}\right| = \frac{1176}{84} = 14$$

23. Let E_1, E_2, E_3 and A be the events defined as follows:

E_1 = bag I is selected

E_2 = bag II is selected

E_3 = bag III is selected

A = one black and one blue ball has been drawn from the selected bag.

As the bags are selected at random, therefore the probability of their selection is equally likely.

$$\therefore \ P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$$

Total number of balls in bag I = 1 + 2 + 3 = 6

$$P(A|E_1) = \frac{{}^2C_1 \times {}^3C_1}{{}^6C_2} = \frac{2 \times 3}{15} = \frac{2}{5}$$

Total number of balls in bag II = 2 + 4 + 1 = 7

$$P(A|E_2) = \frac{{}^4C_1 \times {}^1C_1}{{}^7C_2} = \frac{4 \times 1}{21} = \frac{4}{21}$$

Total number of balls in bag III = 3 + 3 + 2 = 8

$$P(A|E_3) = \frac{{}^3C_1 \times {}^2C_1}{{}^8C_2} = \frac{3 \times 2}{28} = \frac{3}{14}$$

By using law of total probability, we get

$P(A) = P(E_1)\ P(A|E_1) + P(E_2)\ P(A|E_2)$

$$+ P(E_3)\ P(A|E_3)$$

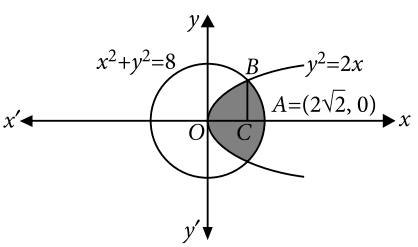
$$= \frac{1}{3} \cdot \frac{2}{5} + \frac{1}{3} \cdot \frac{4}{21} + \frac{1}{3} \cdot \frac{3}{14} = \frac{1}{3}\left[\frac{2}{5} + \frac{4}{21} + \frac{3}{14}\right]$$

$$= \frac{1}{3} \cdot \frac{169}{210} = \frac{169}{630}$$

OR

We have $y^2 = 2x$ ...(i), a parabola with vertex (0, 0) and $x^2 + y^2 = 8$ ...(ii), a circle with centre (0, 0) and radius $2\sqrt{2}$

Let the area of the smaller part of the circle be A_1 and that of the bigger part be A_2. We have to find $\dfrac{A_1}{A_2}$.



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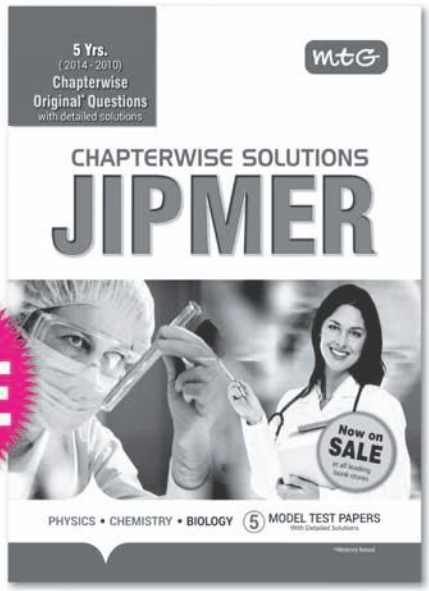

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On solving (i) and (ii), we get $x = 2, -4$

$x = -4$ is not possible as both the points of intersection have the same positive x-coordinate.

Now, $A_1 = 2[\text{Area}(OBCO) + \text{Area}(CBAC)]$

$$= 2\left[\int_0^2 y_1\,dx + \int_2^{2\sqrt{2}} y_2\,dx\right]$$

$$\Rightarrow \quad A_1 = 2\left[\int_0^2 \sqrt{2x}\,dx + \int_2^{2\sqrt{2}} \sqrt{8-x^2}\,dx\right]$$

$$\Rightarrow \quad A_1 = 2\left[\sqrt{2}\cdot\frac{2}{3}x^{3/2}\right]_0^2 + 2\left[\frac{x}{2}\sqrt{8-x^2} + \frac{8}{2}\sin^{-1}\frac{x}{2\sqrt{2}}\right]_2^{2\sqrt{2}}$$

$$= \frac{16}{3} + 2\left[2\pi - \left(2 + 4\times\frac{\pi}{4}\right)\right] = \left(\frac{4}{3} + 2\pi\right) \text{sq. units}$$

Area of circle $= \pi(2\sqrt{2})^2 = 8\pi$ sq. units

Hence, $A_2 = 8\pi - A_1 = 6\pi - \frac{4}{3}$

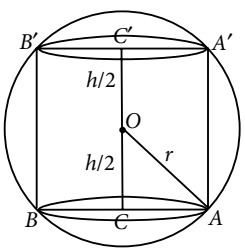
Then, the required ratio is

$$\frac{A_1}{A_2} = \frac{\frac{4}{3} + 2\pi}{6\pi - \frac{4}{3}} = \frac{2 + 3\pi}{9\pi - 2}$$

24. Let h be the height and R be the radius of the base of the inscribed cylinder. Let V be the volume of the cylinder and r be the radius of sphere. Then,

$$V = \pi R^2 h \qquad \ldots(i)$$

From ΔOCA, we have



$$r^2 = \left(\frac{h}{2}\right)^2 + R^2$$

$$\Rightarrow R^2 = r^2 - \frac{h^2}{4}$$

$$\therefore \quad V = \pi\left(r^2 - \frac{h^2}{4}\right)h$$

$$\Rightarrow \quad V = \pi r^2 h - \frac{\pi}{4}h^3$$

$$\Rightarrow \quad \frac{dV}{dh} = \pi r^2 - \frac{3\pi h^2}{4} \text{ and } \frac{d^2V}{dh^2} = -\frac{3\pi h}{2}$$

For maximum or minimum values of V, we must have

$$\frac{dV}{dh} = 0 \Rightarrow \pi r^2 - \frac{3\pi h^2}{4} = 0 \Rightarrow h^2 = \frac{4r^2}{3} \Rightarrow h = \frac{2r}{\sqrt{3}}$$

$$\left[\text{neglecting } h = \frac{-2}{\sqrt{3}}r \because \left.\frac{d^2V}{dh^2}\right|_{h=\frac{-2}{\sqrt{3}}r} > 0\right]$$

Now, $\left(\frac{d^2V}{dh^2}\right)_{h=\frac{2r}{\sqrt{3}}} = -\sqrt{3}\pi r < 0$

Thus, V is maximum when $h = \frac{2r}{\sqrt{3}}$.

Putting $h = \frac{2r}{\sqrt{3}}$ in $R^2 = r^2 - \frac{h^2}{4}$, we obtain

$$R = \sqrt{\frac{2}{3}}r. \qquad \text{[neglecting –ve value of } R\text{]}$$

$\therefore$ The maximum volume of the cylinder is given by

$$V = \pi R^2 h = \pi\left(\frac{2}{3}r^2\right)\left(\frac{2r}{\sqrt{3}}\right) = \frac{4\pi r^3}{3\sqrt{3}}$$

25. We have,

$$\tan^{-1}\left(\frac{x+1}{x-1}\right) + \tan^{-1}\left(\frac{x-1}{x}\right) = \tan^{-1}(-7)$$

$$\Rightarrow \tan^{-1}\left(\frac{1+\frac{1}{x}}{1-\frac{1}{x}\times 1}\right) + \tan^{-1}\left(\frac{x-1}{x}\right) = -\tan^{-1}(7)$$

$$\Rightarrow \tan^{-1}(1) + \tan^{-1}\left(\frac{1}{x}\right) + \tan^{-1}\left(\frac{x-1}{x}\right) + \tan^{-1}7 = 0$$

$$\Rightarrow \pi + \tan^{-1}\left\{\frac{1+7}{1-7}\right\} + \tan^{-1}\left\{\frac{\frac{1}{x}+\frac{x-1}{x}}{1-\frac{x-1}{x^2}}\right\} = 0$$

$$\Rightarrow \quad -\tan^{-1}\frac{4}{3} + \tan^{-1}\left(\frac{x^2}{x^2-x+1}\right) = -\pi$$

$$\Rightarrow \quad \tan^{-1}\left\{\frac{\frac{x^2}{x^2-x+1} - \frac{4}{3}}{1+\frac{4x^2}{3x^2-3x+3}}\right\} = -\pi$$

$$\Rightarrow \quad \frac{3x^2-4x^2+4x-4}{3x^2-3x+3+4x^2} = \tan(-\pi) = -\tan\pi = 0$$

$$\Rightarrow \quad -x^2+4x-4=0$$

$$\Rightarrow \quad x^2-4x+4=0$$

$$\Rightarrow \quad (x-2)^2=0 \quad \Rightarrow \quad x=2$$

26. We have,

$$I=\int \frac{3\cos x+2}{\sin x+2\cos x+3}\,dx$$

Let $3\cos x+2=\lambda(\sin x+2\cos x+3)$
$+\mu(\cos x-2\sin x)+\nu$

Comparing the coefficients of sin x, cos x and constant term on both sides, we get

$\lambda-2\mu=0,\ 2\lambda+\mu=3,\ 3\lambda+\nu=2$

$$\Rightarrow \quad \lambda=\frac{6}{5},\ \mu=\frac{3}{5} \text{ and } \nu=-\frac{8}{5}$$

$$\therefore\ I=\int \frac{\lambda(\sin x+2\cos x+3)+\mu(\cos x-2\sin x)+\nu}{\sin x+2\cos x+3}\,dx$$

$$\Rightarrow \quad I=\lambda\int dx+\mu\int \frac{\cos x-2\sin x}{\sin x+2\cos x+3}dx$$
$$+\nu\int \frac{1}{\sin x+2\cos x+3}dx$$

$$\Rightarrow \quad I=\lambda x+\mu\log|\sin x+2\cos x+3|+\nu I_1,$$

where $I_1=\int \frac{1}{\sin x+2\cos x+3}dx$

Putting, $\sin x=\frac{2\tan(x/2)}{1+\tan^2(x/2)}, \cos x=\frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}$, we get

$$I_1=\int \frac{1}{\frac{2\tan(x/2)}{1+\tan^2(x/2)}+\frac{2(1-\tan^2(x/2))}{1+\tan^2(x/2)}+3}\,dx$$

$$\Rightarrow I_1=\int \frac{1+\tan^2(x/2)}{2\tan(x/2)+2-2\tan^2(x/2)+3(1+\tan^2(x/2))}\,dx$$

$$\Rightarrow \quad I_1=\int \frac{\sec^2(x/2)}{\tan^2(x/2)+2\tan(x/2)+5}\,dx$$

Putting $\tan\frac{x}{2}=t \ \Rightarrow\ \frac{1}{2}\sec^2\frac{x}{2}dx=dt$

$$\Rightarrow \quad I_1=\int \frac{2dt}{t^2+2t+5}=2\int \frac{dt}{(t+1)^2+2^2}$$

$$=\frac{2}{2}\tan^{-1}\left(\frac{t+1}{2}\right)+C=\tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{2}\right)+C$$

Hence, $I=\lambda x+\mu\log|\sin x+2\cos x+3|$
$+\nu\tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{2}\right)+C$

$$\Rightarrow \quad I=\frac{6}{5}x+\frac{3}{5}\log|\sin x+2\cos x+3|$$
$$-\frac{8}{5}\tan^{-1}\left(\frac{\tan(x/2)+1}{2}\right)+C$$

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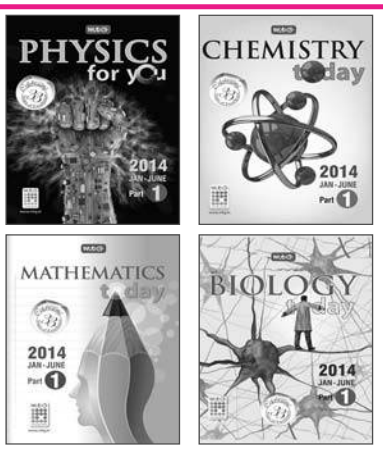

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MATHS MUSING

SOLUTION SET-146

1. (d) : $z = ki, k \in R$

$-(\cos\alpha + i\sin\alpha)k^2 + ki + 1 = 0$

$\Rightarrow \; k^2\cos\alpha = 1, k\sin\alpha = 1.$

Eliminating k,

$$\cos\alpha = \frac{\sqrt{5}-1}{2}, \tan\alpha = \sqrt{\frac{\sqrt{5}+1}{2}}$$

2. (a): $x = e^t \Rightarrow I = \int_0^1 t^3 e^{3t}\,dt = \frac{2}{27}(2e^3+1)$

3. (c): $N = \frac{1}{2}[(-1+3-5+7-.....+19)^2$

$-(1^2 + 3^2 + 5^2 + ... + 19^2)]$

$= -615, |N| = 615$, with digit sum 12

4. (c): $x = 0.\overline{abc}$, $999x = abc$

$999\sum x = 111\cdot 9\cdot 8(0+1+.....+9)$

$\therefore \quad \sum x = 360$, with digit sum 9.

5. (b) : $2ab\cos C = a^2 + b^2 - c^2$

$\therefore \quad \sqrt{3}(x^2+x+1)(x^2-1)$

$= (x^2+x+1)^2 + (x^2-1)^2 - (2x+1)^2$

$\Rightarrow \; (2-\sqrt{3})x^2 + (2-\sqrt{3})x - (1+\sqrt{3}) = 0$

Solving, $x = 1+\sqrt{3}, -(2+\sqrt{3})$

$x = -(2+\sqrt{3}) \;\Rightarrow\; 2x+1 = c < 0 \;\therefore\; x = 1+\sqrt{3}$

6. (a, c): A vector in the plane of $\vec{b}$ and $\vec{c}$ is

$\vec{b} + \lambda\vec{c} = (1+\lambda)\hat{i} + (2+\lambda)\hat{j} - (1+2\lambda)\hat{k}$

$$\sqrt{\frac{2}{3}} = \frac{(\vec{b}+\lambda\vec{c})}{|\vec{a}|}\cdot\vec{a} \;\Rightarrow\; \lambda = 1, -3$$

The vectors are $2\hat{i}+3\hat{j}-3\hat{k}, -2\hat{i}-\hat{j}+5\hat{k}$

7. (b) : $PQ^2 = 2\cos^2\theta + (1+\sin\theta)^2 = f(\theta)$

$f'(\theta) = 0 \;\Rightarrow\; \theta = \frac{\pi}{2}, \; PQ_{max} = 2$

8. (d) : $PQ^2 = 5\cos^2\theta + (1+\sin\theta)^2 = f(\theta)$

$f'(\theta) = 0 \;\Rightarrow\; \sin\theta = \frac{1}{4}, \; PQ_{max} = \frac{5}{2}$

9. (3): CC LL UU AS

$5 = 2+2+1 \Rightarrow \binom{3}{2}\binom{3}{1}\cdot\frac{5!}{2!2!} = 270$ words

$5 = 2+1+1+1 \Rightarrow \binom{3}{1}\binom{4}{3}\cdot\frac{5!}{2!} = 720$ words

$5 = 1+1+1+1+1 \Rightarrow \binom{5}{5}\cdot 5! = 120$ words

Total = 1110 words

Digit sum of $N = 3$

10. (c): P. $t = \frac{\pi}{2}\cos x \;\Rightarrow\; I = \frac{16}{\pi^2}\int_0^{\pi/2} t\sin t\;dt = \frac{16}{\pi^2}$

Q. $$I = \int_0^{\pi} \frac{\sin 2x \sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}dx$$

Replacing x by $\pi - x$, $I = -I \Rightarrow I = 0$

R. Replacing x by $\pi - x$ and adding with I

$\Rightarrow \; 2I = \int_0^{\pi} f(x)dx = \frac{16}{\pi^2} \Rightarrow I = \frac{8}{\pi^2}$

S. Replacing x by $\pi - x$ and adding with I

$\Rightarrow 2I = \pi\int_0^{\pi} f(x)dx = \pi\cdot\frac{16}{\pi^2} \Rightarrow I = \frac{8}{\pi}$

Solution Sender of Maths Musing

SET-146

1.	Sattwik Sadhu	:	W.B.
2.	Chirayata Bhattacharyya	:	W.B.
3.	Khokon Kumar Nandi	:	W.B.
4.	Gouri Sankar Adhikari	:	W.B.
5.	N. Jayanthi	:	Hyderabad

SET-145

1.	Gouri Sankar Adhikari, Rintu Giri, Chandan Pramanik	:	W.B.
2.	Devjit Acharjee	:	W.B.
3.	Shreyam Maity	:	W.B.
4.	Khokon Kumar Nandi	:	W.B.

PRACTICE PAPER 2015

JEE (Main & Advanced) and other PETs

PAPER-1

SECTION-I

MULTIPLE CORRECT CHOICE TYPE

This section contains 10 multiple choice questions. Each question has four choices (a), (b), (c) and (d) out of which ONE or MORE may be correct. [Correct answer 3 marks and wrong answer no negative mark]

1. The solution of $\frac{dy}{dx}=\frac{x^2+y^2+1}{2xy}$ satisfying $y(1)=1$ is

(a) a hyperbola (b) a circle

(c) $y^2=x(1+x)-1$

(d) $(x-2)^2+(y-3)^2=5$

2. If the antiderivative of $\sin^{-1}\sqrt{\frac{x}{x+1}}$ is $x\sin^{-1}\sqrt{\frac{x}{x+1}}-\sqrt{x}+fog(x)+c$, then

(a) $f(x)=\sin^{-1}x,\ g(x)=\sqrt{x}$

(b) $g(x)=\sqrt{x+1},\ f(x)=\tan^{-1}x$

(c) $f(x)=\tan^{-1}x,\ g(x)=\sqrt{x}$

(d) none of these

3. The value of the integral $\int_0^1 e^{x^2}dx$ is

(a) less than e (b) greater than e

(c) less than 1 (d) greater than 1

4. The value of the integral $\int_0^{\pi/4}\frac{dx}{a^2\cos^2x+b^2\sin^2x}$ is

(a) $\frac{1}{ab}\tan^{-1}\left(\frac{b}{a}\right)(a>0,\ b>0)$

(b) $\frac{1}{ab}\tan^{-1}\left(\frac{b}{a}\right)(a<0,\ b<0)$

(c) $\frac{\pi}{4}(a=1,\ b=1)$

(d) $\frac{1}{ab}\tan^{-1}\left(\frac{a}{b}\right)+\frac{1}{ab}$

5. The graph of the function $y=f(x)$ passing through the point (0, 1) and satisfying the differential equation $\frac{dy}{dx}+y\cos x=\cos x$ is such that it is

(a) a constant function

(b) periodic

(c) neither an even nor an odd function

(d) continuous and differentiable for all x

6. The differential equation of all parabolas each of which has a latus rectum $4a$ and whose axes are parallel to x-axis is

(a) of degree 2 and order 1

(b) of order 2 and degree 3

(c) $2a\frac{d^2x}{dy^2}=1$

(d) $2a\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3=0$

7. $\int_0^{\pi/2}\left(\frac{1+\sin 3x}{1+2\sin x}\right)dx$ is

(a) $\int_0^{\pi/2}\left(\frac{1-\cos 3x}{1+2\cos x}\right)dx$

(b) $\int_0^{\pi/2}\left(\frac{\cos 3x+1}{2\cos x-1}\right)dx$

(c) $\frac{\pi}{2}$ (d) 1

By : **Vidyalankar Institute**, Pearl Centre, Senapati Bapat Marg, Dadar (W), Mumbai - 28. Tel.: (022) 24306367

8. The integral of $\frac{1}{\sin^2 x+\tan^2 x}$ must be

(a) $-\frac{1}{2}\left[\tan x+\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\tan x\right)\right]+c$

(b) $-\frac{1}{2}\left[\cot x+\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\tan x\right)\right]+c$

(c) $-\left[\cot x-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\tan x\right)\right]+c$

(d) $-\frac{1}{2}\left[\cot x-\frac{1}{\sqrt{2}}\cot^{-1}\left(\frac{1}{\sqrt{2}}\tan x\right)\right]+c$

9. Orthogonal trajectories of family of parabolas $y^2 = 4a(x + a)$, where a is an arbitrary constant is

(a) $ax^2 = 4cy$ (b) $x^2 + y^2 = a^2$

(c) $y=ce^{-\frac{x}{2a}}$ (d) $axy = c^2$

where c is a constant.

10. $\int\left(\frac{\ln x-1}{(\ln x)^2+1}\right)^2 dx$ is equal to

(a) $\frac{x}{x^2+1}+c$ (b) $\frac{\ln x}{(\ln x)^2+1}+c$

(c) $\frac{x}{(\ln x)^2+1}+c$ (d) $e^x\left(\frac{x}{x^2+1}\right)+c$

SECTION-II

ONE INTEGER VALUE CORRECT TYPE

This section contains 10 questions. Each question, when worked out will result in one integer from 0 to 9 (both inclusive). [Correct answer 3 marks & wrong answer no negative mark]

11. If $f(x) = \max\{2 - x, 2, 1 + x\}$, then $\int_{-1}^{2} f(x)dx =$

12. The area bounded by the curves $y = \ln x$, x-axis and $x = e$ is

13. If $g(x)=\frac{1}{x}\int_{2}^{x}\{3t-2g'(t)\}dt$, then $2g'(2) =$

14. Find the area enclosed by the curve $[x] + [y] = 4$ in the 1st quadrant (where $[\cdot]$ denotes greatest integer function).

15. The order of the differential equation whose general solution is given by $y = C_1 + C_2 \cos x + C_3 + C_4e^{x + C_5}$, where C_1, C_2, C_3, C_4, C_5 arbitrary constants are

16. The value of integral, $\int_{0}^{\pi/2} \sin 2x \tan^{-1}(\sin x)dx$ must be equal to $\frac{\pi}{2}-k$, where k is

17. The value of $\int_{1}^{e^{37}} \frac{\pi\sin(\pi\ln x)}{x}dx$ is

18. The value of $\int_{-2}^{2} |1-x^2|\,dx$ is

19. If $\int_{0}^{\pi/2} f(\sin 2x)\sin x\,dx = A\sqrt{2}\int_{0}^{\pi/4} f(\cos 2x)\cos x\,dx$, then the value of A is

20. The value of $y\sqrt{8}-\frac{1}{3}$ if $1+x^2\frac{dy}{dx}=x(1-y)$, $y(0) = 4/3$ is

PAPER-2

SECTION-I

SINGLE CORRECT OPTION

This section contains 10 multiple choice questions. Each question has four choices (a), (b), (c) and (d) for its answer, out of which ONLY ONE is correct. [Correct answer 3 marks and wrong answer –1 mark]

1. The area bounded by the curves $f(x) = \sin^{-1}(\sin x)$ and $g(x) = [\sin^{-1}(\sin x)]$ in the interval $[0, \pi]$, where $[\cdot]$ is a greatest integer function, is

(a) $\pi/2$ (b) $\left(\frac{\pi}{2}-1\right)^2$

(c) π (d) $\left(\frac{\pi}{4}-1\right)^2$

2. The value of $\int_{0}^{2}[x^2-1]dx$, where $[\cdot]$ denotes the greatest integer function, is given by

(a) $3-\sqrt{3}-\sqrt{2}$ (b) $2-\sqrt{3}$

(c) $4-\sqrt{3}-\sqrt{2}$ (d) none of these

3. $\int_0^{\infty} \frac{x \log x dx}{(1+x^2)^2}$ is equal to

(a) 1 (b) 0

(c) 2 (d) none of these

4. If $A = \int_0^{3\pi/2} \frac{\cos x}{\cos x - \sin x} dx$ and

$B = \int_0^{3\pi/2} \frac{\sin x}{\cos x - \sin x} dx$, then the value of $A + B$ is

(a) $3\pi/2$ (b) $\pi/4$ (c) 0 (d) π

5. The anti-derivative of $f(x) = \frac{1}{3 + 5\sin x + 3\cos x}$, whose graph passes through the point (0, 0) is

(a) $\frac{1}{5}\ln\left|1 - \frac{5}{3}\tan\frac{x}{2}\right|$ (b) $\frac{1}{5}\ln\left|1 + \frac{5}{3}\tan\frac{x}{2}\right|$

(c) $\frac{1}{5}\ln\left|1 + \frac{5}{3}\cot\frac{x}{2}\right|$ (d) none of these

6. $\int \frac{xe^x}{(1+x)^2} dx$ is

(a) $\frac{e^x}{x+1} + c$ (b) $e^x(x + 1) + c$

(c) $-\frac{e^x}{(x+1)^2} + c$ (d) $\frac{e^x}{1+x^2} + c$

7. A solution of $y = 2x\left(\frac{dy}{dx}\right) + x^2\left(\frac{dy}{dx}\right)^4$ is

(a) $y = 2\, c^{1/2}\, x^{1/4} + c$ (b) $y = 2\sqrt{c}x^2 + c^2$

(c) $y = 2\sqrt{c}(x+1)$ (d) $y = 2\sqrt{cx} + c^2$

8. If $\int \tan^7 x dx = f(x) + \log|\cos x|$, then

(a) $f(x)$ is a polynomial of degree 8 in tan$x$

(b) $f(x)$ is a polynomial of degree 5 in tan$x$

(c) $f(x)$ is a polynomial of degree 6 in tan$x$

(d) $f(x) = \frac{\tan^6 x}{6} - \frac{1}{4}\tan^4 x + \frac{1}{2}\tan^2 x + \log|\cos x| + c$

9. If $I_{m,n} = \int \cos^m x \sin nx\, dx$, then $7I_{4,3} - 4I_{3,2} =$

(a) constant (b) $-\cos^2 x + c$

(c) $-\cos^4 x \cos 3x + c$ (d) $\cos 7x - \cos 4x + c$

10. If $f(x)$ is an even and differentiable function, then the value of $\int_{-1}^{1} \left[x^3 f(x) + xf''(x) + 3\right] dx =$

(a) 6 (b) 2 (c) 0 (d) 1

SECTION-II

PARAGRAPH TYPE

This section contains 3 paragraph. Based upon each of the paragraphs 2 multiple choice questions have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which ONLY ONE is correct. [Correct answer 3 marks & wrong answer –1 mark]

Paragraph for Q. No. 11 and 12

Consider the integral $\int \frac{\phi(x)dx}{\sqrt{(ax^2 + bx + c)}}$, where $\phi(x)$ is a polynomial in x.

If $\phi(x)$ is a polynomial of degree n, then there exists a polynomial $f(x)$ of degree $(n - 1)$ and a constant such that

$$\int \frac{\phi(x)dx}{\sqrt{(ax^2 + bx + c)}} = f(x)\sqrt{ax^2 + bx + c} + D\int \frac{dx}{\sqrt{ax^2 + bx + c}}$$

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11. Differentiating both sides with respect to x and multiply by $\sqrt{ax^2+bx+c}$, we get

(a) $\phi(x)=f'(x)(ax^2+bx+c)+\frac{1}{2}(2ax+b)f(x)+D$

(b) $\phi(x)=f'(x)(ax^2+bx+c)-\frac{1}{2}(2ax+b)f(x)+D$

(c) $\phi(x)=\frac{1}{2}f'(x)(ax^2+bx+c)+\frac{1}{2}(2ax+b)f(x)+D$

(d) none of these

12. Now apply this method to evaluate the given integral.

If $\int\frac{(x^3+4x^2-6x+3)dx}{\sqrt{5+6x-x^2}}$

$=(Ax^2+Bx+C)\sqrt{(5+6x-x^2)}+D\sin^{-1}\left(\frac{3-x}{\sqrt{14}}\right)$,

then

(a) $A=\frac{2}{3}$ (b) $B=\frac{9}{5}$

(c) $C=\frac{1}{6}$ (d) $C=\frac{227}{6}$

Paragraph for Q. No. 13 and 14

An even function f is defined and integrable everywhere and is periodic with period 2.

Also, function $g(x)=\int_0^x f(t)dt$ and $g(1)=A$.

13. Function $g(x)$ is

(a) odd (b) even

(c) neither even nor odd

(d) can't be determined

14. Value of $g(2)$ in terms of A is

(a) $2A$ (b) $A/2$

(c) $4A$ (d) $A/4$

Paragraph for Q. No. 15 and 16

Let $f(x)=\frac{2}{x^3+6x^2+11x+6}$

15. Then $\int_0^1 f(x)dx$ is equal to

(a) 3ln3 – 5ln2 (b) 5ln2 – 3ln3

(c) 5ln2 + 3ln3 (d) none of these

16. $f(n)$ is equal to

(a) $\frac{{}^nC_0}{3}-\frac{{}^nC_1}{4}+\frac{{}^nC_2}{5}-....$ to $(n+1)$ terms

(b) $\frac{{}^nC_1}{3}-\frac{{}^nC_2}{4}+\frac{{}^nC_3}{5}-....$ to $(n+1)$ terms

(c) $\frac{{}^nC_3}{3}-\frac{{}^nC_4}{4}+\frac{{}^nC_5}{5}-....$ to $(n+1)$ terms

(d) none of these

SECTION-III

MATCHING LIST TYPE

This section contains 4 questions, each having two matching lists. Choices for the correct combination of elements from List-I and List-II are given as options (a), (b), (c) and (d), out of which one is correct. [Correct answer 3 marks & wrong answer –1 mark]

17. Match the following.

	Column I		Column II
(P)	The equation of curve $y^2(a^2+x^2)=x^2(a^2-x^2)$. The area of a loop of the above curve is	1.	$4a^2$
(Q)	The area of the curve $a^2y^2=x^2(a^2-x^2)$ is	2.	$2a^2\left(1+\frac{\pi}{4}\right)$
(R)	Area contained between the curve $y^2(a-x)=x^2(a+x)$ and its asymptotes is	3.	$a^2\left(\frac{\pi}{2}-1\right)$
(S)	The area enclosed by the parabola $ay=3(a^2-x^2)$ and the x-axis is	4.	$\frac{4}{3}a^2$

	P	Q	R	S
(a)	1	2	3	4
(b)	3	4	2	1
(c)	3	1	2	4
(d)	3	4	1	2

18. Match the following:

Column I		Column II	
(P)	$\int \frac{1}{\sin x - \cos x}\,dx$	1.	$\tan^{-1}(\tan^2 x) + c$
(Q)	$\int \frac{\sin(2x)}{x \ln x}\,dx$	2.	$\ln x - \ln(1+x) + \frac{1}{1+x} + c$
(R)	$\int \frac{e^{\ln\left(1+\frac{1}{x^2}\right)}}{x^2+\frac{1}{x^2}}\,dx$	3.	$\frac{1}{\sqrt{2}} \ln \tan\left(\frac{\pi}{2} - \frac{x}{8}\right) + c$
(S)	$\int \frac{\ln x}{(1+x)^3}\,dx$	4.	$\frac{1}{\sqrt{2}} \tan^{-1} \frac{x^2-1}{x\sqrt{2}} + c$

	P	Q	R	S
(a)	3	4	1	2
(b)	3	1	2	4
(c)	3	1	4	2
(d)	4	1	2	3

19. Match the following:

Column I		Column II	
(P)	$\int \frac{e^x}{x+2}[(1+(x+2)\ln(x+2)]\,dx$	1.	$\sin^{-1}\left(\frac{2x+3}{\sqrt{17}}\right) + c$
(Q)	$\int \sin^2 x \cos^3 x\,dx$	2.	$\frac{x^4}{4} - \frac{x^2}{2} + \frac{1}{2}\ln(x^2+1) + c$
(R)	$\int \frac{dx}{\sqrt{2-3x-x^2}}$	3.	$e^x \ln(x+2) + c$
(S)	$\int \frac{x^5}{x^2+1}\,dx$	4.	$\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + c$

	P	Q	R	S
(a)	3	4	1	2
(b)	4	1	2	3
(c)	1	2	3	4
(d)	3	1	2	4

20. Match the following:

Column I		Column II	
(P)	$\int_0^{\pi/2} \frac{\cos x}{(1+\sin x)(2+\sin x)}\,dx$	1.	$2\left(1-\frac{1}{e}\right)$
(Q)	$\int_0^{41\pi/4} \lvert\cos x\rvert\,dx$	2.	$\ln 4 - \ln 3$
(R)	$\int_0^1 \frac{\ln(1-x)}{x}\,dx$	3.	$20 + \frac{1}{\sqrt{2}}$
(S)	$\int_{1/e}^{e} \lvert\ln x\rvert\,dx$	4.	$-\frac{\pi^2}{6}$

	P	Q	R	S
(a)	2	3	4	1
(b)	1	2	3	4
(c)	3	4	1	2
(d)	2	1	3	4

ANSWER KEYS

PAPER-1

1.	(a, c)	**2.**	(c)	**3.**	(a, d)
4.	(a, b, c)	**5.**	(a, b, d)	**6.**	(c, d)
7.	(a, b, d)	**8.**	(b, d)	**9.**	(c)
10.	(c)	**11.**	(7)	**12.**	(1)
13.	(3)	**14.**	(5)	**15.**	(3)
16.	(1)	**17.**	(2)	**18.**	(4)
19.	(1)	**20.**	(3)		

PAPER - 2

1.	(b)	**2.**	(a)	**3.**	(b)	**4.**	(c)	**5.**	(b)
6.	(a)	**7.**	(d)	**8.**	(d)	**9.**	(c)	**10.**	(a)
11.	(a)	**12.**	(c)	**13.**	(a)	**14.**	(a)	**15.**	(b)
16.	(a)	**17.**	(b)	**18.**	(c)	**19.**	(a)	**20.**	(a)

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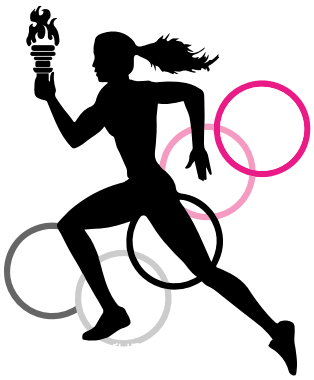


1. ABC is a triangle with $AB \neq AC$. Similar triangles ABD and ACE are drawn outwardly on the sides AB and AC of ΔABC, so that $\angle ABD = \angle ACE$ and $\angle BAD = \angle CAE$. CD and BE meet AB and AC at P and Q respectively. Prove that $AP = AQ$ if and only if $[ABD]\cdot[ACE] = [ABC]^2$, where $[XYZ]$ denotes the area of ΔXYZ.

2. The function f is defined on non-negative integers by, $f(0) = 0$ and $f(2n + 1) = 2f(n)$ for $n \geq 0$, $f(2n) = 2f(n) + 1$ for $n \geq 1$.

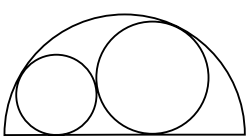
(a) Let $g(n) = f(f(n))$. Show that $g(n - g(n)) = 0$ for all $n \geq 0$.

(b) For any $n \geq 1$, let $r(n)$ be the least integer r such that $f^r(n) = 0$ (where $f^2(n) = f(f(n))$, $f^3(n) = f(f^2(n))$, etc). Compute $\liminf\limits_{n\to\infty} \dfrac{n}{2^{r(n)}}$.

3. In a group of nine mathematicians each speaks at most three languages and any two of them speak at least one common language. Show that at least five of them share a common language.

4. Two externally tangent circles of radii R_1 and R_2 are internally tangent to a semicircle of radius 1, as in the figure. Prove that

$R_1 + R_2 \leq 2(\sqrt{2} - 1)$.



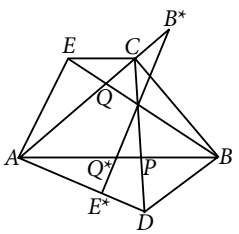
5. In an equilateral triangle ABC (of side length 2) consider the incircle Γ.

(a) Show that for all points P of Γ, $\overline{PA}^2 + \overline{PB}^2 + \overline{PC}^2 = 5$.

(b) Show that for all points P of Γ, it is possible to construct a triangle of sides PA, PB, PC, with area $\sqrt{3}/4$.

SOLUTIONS

1.



Reflecting B, Q and E around the internal bisector of $\angle BAC$, we get collinear points B^*, Q^* and E^* on the lines AC, AB and AD respectively. Now, since A will lie between P and B and between Q and C if and only if $\angle BAE = \angle DAC > 180°$, $AP = AQ$ if and only if $P = Q^*$, *i.e.*

$AP = AQ \Leftrightarrow B^*$, P and E^* are collinear. ....(1)

From $AB \neq AC$, it follows that $B^* \neq C$ and $E^* \neq D$. Let $AB = c$, $AC = b$, and

$AD : AB = AE : AC = x : 1$,

so that $AE^* = AE = xb$ and $AD = xc$. Then from (1), and according to Menelaos' Theorem applied to ΔACD, $AP = AQ$ if and only if

$$1 = \frac{AE^*\cdot DP\cdot CB^*}{DE^*\cdot CP\cdot AB^*} = \frac{xb\cdot DP\cdot(c-b)}{(xc-xb)\cdot PC\cdot c} = \frac{b\cdot DP}{c\cdot PC},$$

that is,

$[ACE] : [ABD] = b^2 : c^2 = (PC)^2 : (DP)^2$

$= [ABC]^2 : [ABD]^2$

that is,
$[ABD]\cdot[ACE] = [ABC]^2$.

2. (a) Most of the solution can be worked out suitably in binary notation. To prepare for this, we first prove the following lemma.

LEMMA : Let $L(n) = 2^{\lfloor \log_2(2n) \rfloor}, n \geq 1$,

where $\lfloor x \rfloor$ denotes the greatest integer $\leq x$. Then $f(n) = L(n) - n - 1$.

Proof : Since

$$1 + \lfloor \log_2(2n) \rfloor = \lfloor 1 + \log_2(2n) \rfloor = \lfloor \log_2(4n) \rfloor$$

we have $2L(n) = L(2n)$. Furthermore, we claim that $L(2n + 1) = L(2n)$. Suppose $\lfloor \log_2(2n) \rfloor < \lfloor \log_2(2n+1) \rfloor$. Then letting $\lfloor \log_2(2n) \rfloor = k$, we have $\log_2(2n) < k + 1 \leq \log_2(2n + 1)$ which implies $2n < 2^{k+1} \leq 2n + 1$ or $n < 2^k \leq n + 1/2$, clearly an impossibility.

Since $f(1) = 0$ and $L(1) = 2$, $f(n) = L(n) - n - 1$ holds for $n = 1$.

Assume the formula holds up to $2n - 1$ for some $n \geq 1$. Then

$$f(2n + 1) = 2f(n) = 2L(n) - 2n - 2$$
$$= L(2n) - 2n - 2 = L(2n + 1) - (2n + 1) - 1 \text{ and}$$
$$f(2n) = 2f(n) + 1 = L(2n) - 2n - 1.$$

This completes the proof of the lemma.

Since $g(0) = f(f(0)) = f(0) = 0$, $g(0 - g(0)) = 0$. Assume henceforth that $n \geq 1$. The binary representation of n can be written in the form

$$n = \underbrace{1.....1}_{s}\underbrace{0.....0}_{t}\underbrace{1*.....*}_{u},$$

where each $*$ is either 0 or 1, and $s \geq 1$, t, $u \geq 0$. Since $L(n) - 1$ is just a string of 1's in binary notation, and since $n + f(n) = L(n) - 1$ by the lemma, we see that f acts on n by interchanging all 1's and 0's in the binary expansion of n. In other words,

$$f(n) = \underbrace{1.....1}_{t}\underbrace{0*'.....*'}_{u}$$

where each $*'$ is $1 - *$. Thus,

$$g(n) = f(f(n)) = \underbrace{1*.....*}_{u};$$

i.e., $g(n)$ is the original final block of u digits in n. As a result,

$$n - g(n) = \underbrace{1.....1}_{s}\underbrace{0.....0}_{t+u}.$$

Since the last expression contains no final block of digits after the initial block of 1's and 0's, we have $g(n - g(n)) = 0$ for all $n \geq 1$.

(b) The answer is 2/3. To see this, it is necessary to examine more closely the blocks of binary digits of n. Since

$$n = \begin{cases} \underbrace{1.....1}_{s_1}\underbrace{0.....0}_{s_2}....\underbrace{1.....1}_{s_m} & \text{if } m \text{ is odd,} \\ \underbrace{1.....1}_{s_1}\underbrace{0.....0}_{s_2}....\underbrace{0.....0}_{s_m} & \text{if } m \text{ is even,} \end{cases}$$

and since f always wipes out the leading block of 1's, it is clear that $r(n) = m$. Since n has atleast $r(n)$ digits and $2^{r(n)}$ has $1 + r(n)$ digits, $n/2^{r(n)}$ will exceed 1 unless n has exactly $r(n)$ digits; *i.e.*, unless n is in the sequence (in base 2) 1, 10, 101, 1010, 10101,.... . So the limit inferior of $n/2^{r(n)}$ is the limit of the sequence (in base 2)

$$\frac{1}{10}, \frac{10}{10^2}, \frac{101}{10^3}, \ldots\ldots, \text{ which is (in base 10)}$$

$$\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \ldots\ldots = \frac{2}{3}.$$

3. If any one of them speaks less than three languages then by the pigeon hole principle one of the languages he (or she) speaks should be spoken by four others and we are done. Suppose now that each speaks three languages. Three cases arise:

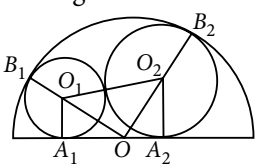
(i) Two mathematicians have all the three languages in common. Here again we are through since one of the three languages should be spoken by at least three of the remaining seven mathematicians.

(ii) Some two mathematicians have two languages in common, say, M_1 and M_2 have L_1 and L_2 in common. Let L_3 and L_4 be the third languages of M_1 and M_2 respectively. If there is one more mathematician having L_1 and L_2 as two of his languages then we are done as follows. Suppose M_3 speaks (L_1, L_2, L_5). Of the remaining six mathematicians if there are

more than two who speak L_1 or L_2 then we are through, if not, there are four mathematicians who do not speak L_1 or L_2. But then these four are forced to speak L_3, L_4, L_5 to be able to converse with M_1, M_2, M_3 and so there will be five mathematicians who speak L_3, L_4 and L_5. So suppose only two have L_1 and L_2 common. Consider the pair (L_3, L_4). If some three mathematicians speak both these languages we are done as before; if not, there are five who do not speak L_3 and L_4 simultaneously. But then these five should speak either L_1 or L_2 to be able to converse with both M_1 and M_2 which in turn implies that either L_1 or L_2 is spoken by at least five mathematicians.

(iii) Any two of the nine mathematicians have exactly one language in common. Let M_1 speak (L_1, L_2, L_3). The remaining eight have to speak either L_1 or L_2 or L_3 and so one of these three languages has to be spoken by at least three more mathematicians, say L_1 is spoken by M_2, M_3, M_4 also. We will show that L_1 is spoken by all the nine mathematicians. Suppose not; say M_5 has L_2 in common with M_1. Now M_5 has to speak to each of M_2, M_3, M_4 in a different language other than L_2 (since any two mathematicians have only one common language) which is not possible as no mathematician speaks more than three languages and we are done.

4. Let O_1, O_2 and O denote the centres of the circles, and let A_1, A_2, B_1 and B_2 denote the points of tangency of these circles with the semicircle, as shown in the diagram.



Then $O_1O_2 = R_1 + R_2$, $O_1A_1 = R_1$, and $O_2A_2 = R_2$, so

$$A_1A_2 = \sqrt{(O_1O_2)^2 - (O_1A_1 - O_2A_2)^2}$$

$$= \sqrt{(R_1 + R_2)^2 - (R_1 - R_2)^2} = 2\sqrt{R_1R_2}.$$

Also $OB_1 = 1$, $OB_2 = 1$, $O_1B_1 = R_1$ and $O_2B_2 = R_2$, so that $OO_1 = 1 - R_1$ and $OO_2 = 1 - R_2$.

Therefore

$$A_1A_2 = OA_1 + OA_2$$

$$= \sqrt{(OO_1)^2 - (O_1A_1)^2} + \sqrt{(OO_2)^2 - (O_2A_2)^2}$$

$$= \sqrt{(1-R_1)^2 - R_1^2} + \sqrt{(1-R_2)^2 - R_2^2}$$

$$= \sqrt{1-2R_1} + \sqrt{1-2R_2}.$$

Thus

$$\sqrt{1-2R_1} + \sqrt{1-2R_2} = 2\sqrt{R_1R_2}.$$

Squaring, then dividing each term by 2 and rearranging the terms, we get

$$\sqrt{(1-2R_1)(1-2R_2)} = 2R_1R_2 + R_1 + R_2 - 1.$$

Squaring both sides and simplify

$$8R_1R_2 = (2R_1R_2 + R_1 + R_2)^2, \qquad ...(1)$$

So, $2\sqrt{2R_1R_2} = 2R_1R_2 + R_1 + R_2$.

Thus

$$R_1 + R_2 = 2\sqrt{R_1R_2}(\sqrt{2} - \sqrt{R_1R_2}) \qquad ...(2)$$

$$\le (R_1 + R_2)(\sqrt{2} - \sqrt{R_1R_2})$$

and therefore $\sqrt{R_1R_2} \le \sqrt{2} - 1$.

Now consider the function $f(x) = 2x(\sqrt{2} - x)$ $f(x)$ is increasing on the interval $(0, 1/\sqrt{2})$ since $f'(x) = 2\sqrt{2} - 4x > 0$ for x in the interval.

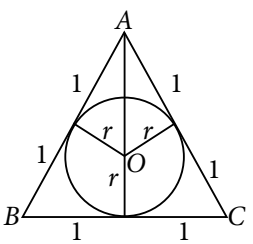
Since $0 < \sqrt{R_1R_2} \le \sqrt{2} - 1 < 1/\sqrt{2}$ and $R_1 + R_2 = f(\sqrt{R_1R_2})$ from (2), $R_1 + R_2$ attains its maximum when $\sqrt{R_1R_2} = \sqrt{2} - 1$. Hence

$$R_1 + R_2 \le 2(\sqrt{2} - 1)[\sqrt{2} - (\sqrt{2} - 1)]$$

$$= 2(\sqrt{2} - 1).$$

Equality holds when $R_1 = R_2$.

5.



Take O as origin, with the x-axis parallel to BC, and y-axis along OA. Then the incircle has radius $r = 1/\sqrt{3}$ and A, B, C have coordinates

$$A\left(0,\frac{2}{\sqrt{3}}\right), B\left(-1,\frac{-1}{\sqrt{3}}\right), C\left(1,\frac{-1}{\sqrt{3}}\right).$$

A point on the incircle has coordinates parameterized by θ, $0 \le \theta < 2\pi$ given by $P\left(\frac{1}{\sqrt{3}}\cos\theta, \frac{1}{\sqrt{3}}\sin\theta\right)$.

(a) Let $c = \cos\theta$, $s = \sin\theta$. Then

$$AP^2 + BP^2 + CP^2$$
$$= \left[\frac{1}{3}c^2 + \frac{1}{3}(2-s)^2\right] + \left[\frac{1}{3}(c+\sqrt{3})^2 + \frac{1}{3}(s+1)^2\right]$$
$$\left[\frac{1}{3}(c-\sqrt{3})^2 + \frac{1}{3}(s+1)^2\right]$$
$$= \frac{1}{3}[c^2 + 4 - 4s^2 + s^2 + c^2 + 2\sqrt{3}c + 3 +$$
$$s^2 + 2s + 1 + c^2 - 2\sqrt{3}c + 3 + s^2 + 2s + 1]$$
$$= \frac{1}{3}[3(c^2 + s^2) + 12] = 5$$

since $c^2 + s^2 = 1$.

(b) Now, set $x = AP = \frac{1}{\sqrt{3}}\sqrt{5-4s}$,

$y = BP = \frac{1}{\sqrt{3}}\sqrt{5+2s+2\sqrt{3}c}$, and

$z = CP = \frac{1}{\sqrt{3}}\sqrt{5+2s-2\sqrt{3}c}$.

By reflection and rotational geometry the distances AP, BP, CP will be a permutation of those obtained when $\pi/6 \le \theta \le \pi/2$, so that $\frac{1}{\sqrt{3}} \le x \le 1$.

Also

$$y = \frac{1}{\sqrt{3}}\sqrt{5+4\left(\frac{1}{2}s + \frac{\sqrt{3}}{2}c\right)} = \frac{1}{\sqrt{3}}\sqrt{5+4\cos\left(\theta - \frac{\pi}{6}\right)}$$

so that $\frac{\sqrt{7}}{\sqrt{3}} \le y \le \sqrt{3}$ and

$$z = \frac{1}{\sqrt{3}}\sqrt{5+4\left(\frac{1}{2}s - \frac{\sqrt{3}}{2}c\right)} = \frac{1}{\sqrt{3}}\sqrt{5+4\sin\left(\theta - \frac{\pi}{3}\right)}$$

so that $\frac{\sqrt{5}}{\sqrt{3}} \le z \le \frac{\sqrt{7}}{\sqrt{3}}$. Thus

$$(x+y)_{\min} = \frac{1+\sqrt{7}}{\sqrt{3}} > z_{\max} = \frac{\sqrt{7}}{\sqrt{3}}$$

and $(y+z)_{\min} = \frac{\sqrt{7}+\sqrt{5}}{\sqrt{3}} > x_{\max} = 1$

and $(z+x)_{\min} = \frac{1+\sqrt{5}}{\sqrt{3}} > y_{\max} = \sqrt{3}$,

and x, y, z (*i.e.* AP, BP, CP) can form the sides of a triangle. From Heron's formula the area, F, of this triangle is given by

$$F^2 = \frac{1}{2}(x+y+z)\frac{1}{2}(-x+y+z)\cdot\frac{1}{2}(x-y+z)\cdot$$
$$\frac{1}{2}(x+y-z)$$
$$= \frac{1}{16}[(y+z)^2 - x^2][x^2 - (y-z)^2]$$
$$= \frac{1}{16}\left[\frac{1}{3}(5+2s+2\sqrt{3}c+5+2s-2\sqrt{3}c-5+4s)\right.$$
$$\left.+2\cdot\frac{1}{3}(5+2s+2\sqrt{3}c)^{1/2}(c+2s-2\sqrt{3}c)^{1/2}\right]$$
$$\cdot\left[\frac{1}{3}(5-4s-5-2s-2\sqrt{3}c-5-2s+2\sqrt{3}c)\right.$$
$$\left.+2\cdot\frac{1}{3}(5+2s+2\sqrt{3}c)^{1/2}(5+2s-2\sqrt{3}c)^{1/2}\right]$$
$$= \frac{1}{48}[(5+8s)+2\sqrt{(5+2s)^2-12c^2}]$$
$$[-(5+8s)+2\sqrt{(5+2s)^2-12c^2}]$$
$$= \frac{1}{48}[100 + 80s + 16s^2 - 48c^2 - 25 - 80s - 64s^2]$$
$$= \frac{1}{48}[75-48] = \frac{9}{16}, \text{ since } c^2 + s^2 = 1.$$

Thus $F = \frac{3}{4}$.

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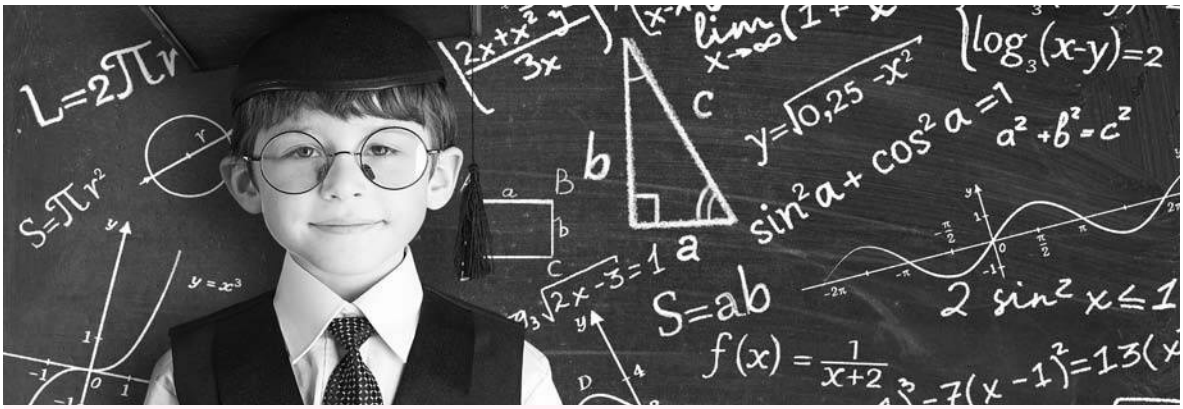
Vital Problem-Solving Skills for Students



Pushing towards growth : Students need more than just technical education to survive today. Problem solving abilities fill that void and help students gain better chances of survival, says Vintu Augustine

Our lives are replete with daily hassles and stressful events. From the morning crossword puzzle to retrieving keys from a locked car, we face manifold difficulties. It becomes extremely difficult to search for solutions for all such matters. The failure to arrive at a solution to a problem puts us into despair and makes us demotivated. The result is stress and disappointment.

Similarly, students also face a number of obstacles in their daily lives. Peer pressure, extreme competition and lack of opportunities make their lives hard and impair their abilities. No matter how small or how big the stressors are, they do have a psychological effect on the young minds. Sure, everyone is born with some talents and abilities in this world but daily hassles and problematic events test every aspect of an individual. Only talent doesn't matter anymore. You need to acquire the abilities to handle problematic situations and emerge successful. To ensure students have a well-rounded personality and survive in this world, problem solving abilities come into the picture. It is a topic of great significance especially in areas like counselling, clinical psychology, social work, health care and so on.

What is problem solving?

Problem solving refers to active efforts to discover what must be done to achieve a goal that is not readily available. Obviously, if a goal is readily attainable, there isn't a problem. But in problem solving situations, one must go beyond the information given to overcome obstacles and reach the goal. Problem solving has been defined as an overt or cognitive process that makes available a variety of potentially effective response strategies for coping with problematic situations. The problem solving skills of a student depends on his critical personal resources for dealing with stressors.

Problem solving is further defined as the complex inter-play of cognitive, affective, and behavioural processes for the purpose of adapting to internal or external demands or challenges.

Effective problem solvers

It has been suggested for some time that ineffective problem solving results in stressful outcomes and psychological maladjustment. It makes sense that effective problem solvers are flexible, adaptable, and are able to develop

suitable methods to solve problems and reach personal goals. Indeed, in the last few years, there has been mounting evidence that problem solving or coping does play a role in adaptive responses to stress. For example, problem solving has been related to depression, hopelessness, suicidal ideation, physical health, help seeking, career planning and academic performance.

Relation to academic performance

Problem solving is very relevant for educators, school psychologists, and student affairs professionals. Educators are often interested not only in imparting knowledge about specific topics but also in increasing students' problem-solving abilities; thus, the advent of problem-based learning has made problem solving central to the educational process. Likewise, student affairs professionals are often interested in psycho-educational programming to impart specific skills (assertiveness), as well as generic problem-solving skills for preventive purposes. In all, problem solving is a topic that has a great deal of applicability for a wide array of practitioners as they work to increase the problem-solving effectiveness of a broad range of people like children, adolescents, college students, adults, and older adults.

Students who promote problem-based learning as an learning approach are expected to be more successful in academic performance and to be better prepared for self-directed, lifelong learning. Most researchers see problem solving as consisting of a sequence of activities, including problem orientation, generation of alternatives, selection of strategies, and evaluation of outcomes. Theoretical accounts of problem solving postulates specific cognitive behavioural skills as important aspect involved in academic achievement.

Effective problem solving can decrease negative affective states by enabling a student more effectively managing the problems. In addition to the link with problems, there is a large body of empirical research demonstrating the relation between problem solving and other variables. Studies have found problem solving to be related to mathematical achievement, verbal and general reasoning ability, spatial ability, field independence, divergent thinking, positive attitudes, and resistance to distraction.

Coping with stress

In a given academic year, a student faces a lot of problems. As the end of a semester approaches, he will have to engage in quite a balancing act in meeting deadlines and fulfilling requirements. They will have papers to write, presentations to give, and exams to prepare for. They may have to maintain a B average to qualify for scholarship or else an A average to sustain parents' support. Given the socio-economic and technological advancements, life in today's world is becoming more and more complex with excess of problems. And the responsibility of school authorities and teachers especially has become increasingly demanding to develop scientific attitude and skills in students to cope with the issues of the time.

The procedure of overcoming difficulties or barriers which interfere with the satisfaction of wants and fulfillment of goals is called problem solving. The nature of approach to problems varies from person to person. It also depends on the difficulty of the problem and the ability of the solvers.

Animals solve their problems by habitual behaviour or by following trial and error method. At the higher level of evolution, the animals also solve their problems by insight. In human beings, reasoning is the most important method of problem solving.

Augmenting performance

We live in an era, when education has taken a commercial turn and there is a renewed thrust on enhancing the performance of students. This can be achieved only by providing quality education. To enhance quality, it is imperative that we assess students' problem solving skills and how that create an impact in their academic performance.

Towards this end, it is vital that we form strategies or prepare intervention programs and necessary tools to improve problem solving abilities of students. Such an empirical research should serve as a fundamental template in syllabus formulation and curriculum design.

It is important to recognise problem solving ability of students as a tool kit to academic performance and in addition, it may become an important predictor of managing daily chaos of life. Therefore, students' life is the perfect time to assess the problem solving ability and determine if there is a need to improve it before they get in to their career life. A better understanding of students' problem solving abilities and training them with strategies to improve their capacities in this regard will definitely go a long way in determining the quality of education today.

Courtesy: Deccan Herald

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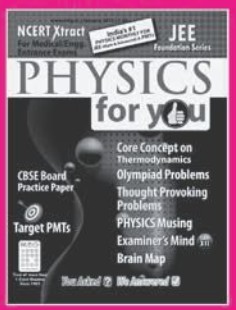

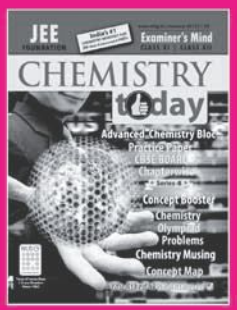

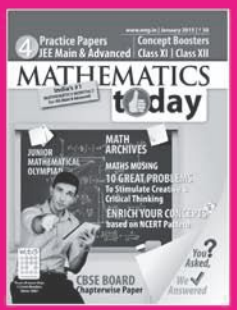

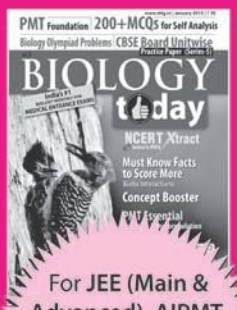

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